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Mean-Distortion Risk Measure Portfolio Optimization under the Distribution Uncertainty

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Abstract: *In this paper, we introduce a universal framework for mean-distortion robust risk measurement and portfolio optimization. We take accounts for the uncertainty based on Gelbrich distance and another uncertainty set proposed by Delage & Ye. We also establish the model under the constraints of probabilistic safety criteria and compare the different frontiers and the investment ratio to each asset. The empirical analysis in the final part explores the impact of different parameters on the model results.*

Keywords: *Robust optimization, Distortion risk measure,Probabilistic safety criteria.*

I. Introduction

In today's world, the financial system is the foundation for the sustained and stable development of the market economy. Financial risks, as an inevitable product of rapid economic development, have become the normal accident. In order to prevent systemic financial risks and maintain the stability of the world financial system, relevant financial institutions must strengthen their awareness of effective risk identification and improve their ability to manage risks. From a financial perspective, the risk of financial assets measures the return or potential losses of investing in different assets, as well as the volatility of assets, which is the degree of deviation from the mean.The "big bang" of financial risk management is the birth of modern portfolio theory, which is a milestone in the history of financial development. Its founder, Markowitz [1], published "Portfolio Selection" in 1952 and won the Nobel Prize in Economics 40 years later. The mean variance portfolio model is a mathematical technique used to develop the optimal combination of portfolio assets under specific risk measures. Its core is a statistical analysis, and its optimal selection is determined by the historical returns of asset classes and the correlation between these returns and other asset returns. Once the mean variance criterion was proposed, it was widely used by institutional investors such as mutual funds.Based on the shortcomings of risk sensitivity measurement indicators in traditional risk measurement, J.P. Morgan [2] proposed a new risk measurement VaR (Value-at-Risk) method in 1994 to meet the needs of his banking business, which was quickly promoted as an industry standard. However, Var does not satisfy subadditivity, which can not explain the nature of risk reduction in diversified investments.

Then Artzner et al. [3] (1999) proposed the concept of Coherent risk measure. They believed that a well-defined risk measure should meet the four axioms of monotonicity, homogeneity, translation invariance and subadditivity. Then Delbaen and Hochachule [4] extended the finite probability space required by the consistent risk measure theory to any probability space, and linked the consistent risk measure with game theory and distortion probability measure. These results are illustrated with corresponding examples. Then Acerbi et al in 2002 [5] put forward the theory of spectral risk measures, which requires that risk measures not only be consistent risk measures, but also have a good risk spectral density to characterize the risk aversion of investors.

In this paper, we mainly focus in the distortion risk measure. This measure was firstly introduced by Wang in 1996 [6] when he calculated the insurance premiums by converting cumulative distribution functions. Also, Wang in 2000 [7] proposed a class of distortion function operators based on normal cumulative distribution, which took into account both assets and liabilities in the pricing formula, linked the CAPM model, and restored the Black Scholes option pricing formula, but this pricing was mainly based on insurance risk. In 2012, Li Jun [8] studied the properties of several specific distortion risk measures within the framework of uncertain distribution, and based on this, used hybrid intelligent algorithms to calculate the mean risk model. In the same year, Feng and Tan [9] proposed the Consistent Distortion Risk Measure (CDRM) based on CVaR's portfolio model and applied it to optimization, comparing the optimal investment portfolios of different transformation measures. Cai and Wang [10] studied the tail subadditivity of distortion risk measures in 2017. They proposed multiple risk measures (MTD) and their properties. Finally, these risk measures were applied to the capital allocation model of venture capital portfolios.

At the same time, many scholars study the robust portfolio optimization based on different kinds of risk measure. In this paper, we follow the concept of Gelbrich distance which was introduced by Gelbrich in 1990 [11] and define the Gelbrich ambiguity set as the family of all asset return distributions with a given structure whose mean-covariance paris reside in a Gelbrich ball around an empirical mean-covariance pair estimated from sample data. Also, we introduced another ambiguity set which was forward by Delage and Ye in 2010 [12]. They use mean and covariance to construct a cone-constrained uncertainty set whose covariance less than specific times sample data's covariance.

In general, the constraint of portfolio investment models is the returnconstraint, but there are also other constraints, such as the safety criterion first proposed by Roy [13] in 1952.Subsequently, Pyle and Turnovsky[14] compared the results of the Markowitz mean variance model based on safety criteria with the traditional expected utility maximization model. On this basis, kataoka and Telser[15] proposed different security criteria. Although the loss objective function is consistent, the optimized objects are different.

II. Problem Statement

In the classic portfolio investment models, it is usually assumed that the market is a complete market. But in real world, we can only observe incomplete information in the market. The decisions made by the investors often rely on the unknown distributions and parameters in the model that also called uncertainty.For example, the distribution of the return shows a fat tail at the tip, so it does not obey the normal distribution. The uncertainty of this distribution is also called ambiguity. Therefore, many scholars have studied portfolio models under ambiguity sets, resulting in a series of distribution uncertainty robust models.Delage and Ye [12] studied the different kinds of uncertainty sets. Chaoui et al. [16] in 2003 discussed the worst-case value-at-risk and robust portfolio optimization. Zhuet al. [17] in 2009 studied worst-case conditional value-at-risk with application to robust portfolio management. Then Kanget al. [18] established data-driven robust mean-CVaR portfolio selection model under distribution ambiguity.

Our research firstly establishes the robust mean-distortion risk measure portfolio selection model and considers the frontier and best investment ratios of each asset under the condition that the moments of return satisfy the constraint of Gelbrich distance. Then inspired by the Kang et al. [18], in which they established the model based on the parameter θ that is the index of ambiguity attitude. The new detaile $\mathrm{d}\theta-\mathrm{robustmodel}$ is as below

$$\min \ (1-\theta)\inf_{Q\in\mathcal{T}}\rho_Q(-x^T\xi)+\theta\sup_{Q\in\mathcal{T}}\rho_Q(-x^T\xi)$$
$$s.t.\ (1-\theta)\sup_{Q\in\mathcal{T}}E_Q\big(R(x,\xi)\big)+\theta\inf_{Q\in\mathcal{T}}E_Q\big(R(x,\xi)\big)\ge\phi$$

III. Distortion Risk Measure

Wang proposed a pricing method using proportional risk transformation when researching premium pricing in 1996, which involved the theory of distortion risk measurement. The distortion risk measure utilizes the distortion function proposed by Yaari in 1987. Although the concept of distortion risk measurement originated from insurance, due to the relationship between insurance and investment risk, distortion risk measurement has also begun to be used for investment environment and portfolio selection issues (such as Van der Hoek and Sherris (2001)).

1.1 Definition of the Distortion risk measure

Definition 1 If $g:(0,1)\to(0,1)$is anon-decreasing function and satisfies$g(0)=0$,$g(1)=1$(g is also called distortion function), the distribution function of$X\in\mathcal{X}$is $F_X(x)$. Then the distortion risk measure is given by

$$\rho_g(X)=-\int_{-\infty}^{0}\big[1-g\big(1-F_X(x)\big)\big]dx+\int_{0}^{\infty}g\big(1-F_X(x)\big)dx\ \#(1)$$

and at least one integral is finite.

When we define the cumulative distribution function (also known as the survival function)$S_X(x)=1-F_X(x)=P(X>x)$,$\rho_g(X)$can also be represented as

$$\rho_g(X)=-\int_{-\infty}^{0}\big[1-g\big(S_X(x)\big)\big]dx+\int_{0}^{\infty}g\big(S_X(x)\big)dx\ \#(2)$$

In some cases, such as issues related to insurance or capital requirements, it can be assumed that the random variable $X\in\mathcal{X}$ is non negative, then

$$\rho_g(X)=\int_{0}^{\infty}g\big(S_X(x)\big)dx\ \#(3)$$

Usually, assuming that the distribution function of the distortion risk g and variable X are independent of each other, the distortion risk measure represents the expectation of a new random variable with a reweighted probability. By changing the probability of risk tolerance and assigning higher probability weights to high-risk events while keeping the loss distribution function unchanged, investors can adjust their subjective probability of tail risk through the distortion function, indicating their aversion to risks at different positions.

The distortion risk measure satisfies the following properties:

Proposition 1 Monotonicity. Suppose $X,Y\in\mathcal{X}$, if $X\le Y$, then$\rho_g(X)\le\rho_g(Y)$.

Proposition 2 Positive homogeneity. For distortion risk measure ρ_g, Let $X\in\mathcal{X}$,$\lambda\ge0$, then we have$\rho_g(\lambda X)=\lambda\rho_g(X)$

Proposition 3 Translation invariance.For distortion risk measure ρ_g, Let $X\in\mathcal{X}$, then we have$\forall c\in R$: $\rho_g(X+c)=\rho_g(X)+c$

Proposition 4 Distortion risk measures are sub-additive if and only if the distortion function$g$is concave. That is $\rho_g(X+Y)\le\rho_g(X)+\rho_g(Y)$

Proposition 5 For a given distortion function g,the distortion risk measure$\rho_g(X)$is coherent risk measure if and only if that $\rho_g(X)$can be represented as

$$\rho_g(X) = \int_{\alpha=0}^{1} \omega(\alpha)\phi_\alpha d\alpha \#(4)$$

Where $\omega : [0,1] \mapsto [0,1], \omega(\alpha) = g'(x)|_{x=1-\alpha}$ and $\phi_\alpha(X)$ is $CVaR_\alpha(x,P)$.

This proposition indicates that coherent risk measure can be expressed as the convex combination of $CVaR_\alpha$. The proof of this see propositioncan be found in Feng and Tan's article.

1.2 Examples of the Distortion risk measure

Example 1 Suppose $X \in \mathcal{X}$, $\alpha \in (0,1)$, and

$$g(x) = \begin{cases} 0 & if\ 0 \le x \le 1-\alpha \\ 1 & if\ 1-\alpha \le x \le 1 \end{cases}$$

Then VaR_α can be expressed as a distortion risk measure, that is

$$VaR_\alpha(X) = \rho_g(X)$$

Example 2 Suppose $X \in \mathcal{X}$, $\alpha \in (0,1)$

$$g(x) = min\left(\frac{x}{1-\alpha}, 1\right)$$

If $x \in [0,1]$, then $CVaR_\alpha(x,P)$ can be expressed as a distortion risk measure, that is

$$CVaR_\alpha(x,P) = \rho_g(X)$$

IV. Robust model under Gelbrich uncertainty set

This section considers the Gelbrichambiguity set defined on the Gelbrich distance, where the distribution of returns is uncertain, but its first-order moment mean and second-order moment variance have certain limitations.

In this paper, we suppose that there are n risky assets in the market. The return on the portfolio is $R(x,\xi) = x^T\xi$. The ratio of investment in the variousrisk asset can be represented as $x = (x_1, x_2, \dots, x_n)^T$, and the return is $\xi = (\xi_1, \xi_2, \dots, \xi_n)$.

1.3 Gelbrich Uncertainty Set

Definition 2[19]Gelbrich distance: The Gelbrich distance between two mean-variance paris $(\mu_1, \Sigma_1), (\mu_2, \Sigma_2) \in R^n \times \mathbb{S}_+^n$ can be given by

$$G\big((\mu_1,\Sigma_1),(\mu_2,\Sigma_2)\big) = \sqrt{\|\mu_1-\mu_2\|^2 + Tr\left[\Sigma_1+\Sigma_2-2\left(\Sigma_2^{\frac{1}{2}}\Sigma_1\Sigma_2^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]} \#(5)$$

We can proof that Gelbrich distance non-negative, symmetric and sub-additive. Only when $(\mu_1, \Sigma_1) = (\mu_2, \Sigma_2)$, this distance equals to 0.

Definition 3[19]Gelbrich ambiguity set: suppose

$$\mathcal{U}_\pi(\hat{\mu},\hat{\Sigma}) = \left\{(\mu,\Sigma) \in R^n \times \mathbb{S}_+^n : G\left((\mu,\Sigma),(\hat{\mu},\hat{\Sigma})\right) \le \pi\right\} \#(6)$$

Then Gelbrichambiguity set is given by

$$\mathcal{G}_\pi(\hat{\mu},\hat{\Sigma}) = \left\{\mathbb{Q} \in \mathcal{S} : \left(E_\mathbb{Q}[\xi], E_\mathbb{Q}\left[(\xi - E_\mathbb{Q}[\xi])(\xi - E_\mathbb{Q}[\xi])^T\right]\right) \in \mathcal{U}_\pi(\hat{\mu},\hat{\Sigma})\right\} \#(7)$$

Proposition 6 For the set $\mathcal{U}_\pi(\hat{\mu},\hat{\Sigma}) = \left\{(\mu,\Sigma) \in R^n \times \mathbb{S}_+^n : G\left((\mu,\Sigma),(\hat{\mu},\hat{\Sigma})\right) \le \pi\right\}$ and $\mathcal{V}_\pi(\hat{\mu},\hat{\Sigma}) = \{(\mu, M) \in R^n \times \mathbb{S}_+^n : (\mu, \mathrm{M} - \mu\mu^T) \in \mathcal{U}_\pi(\hat{\mu},\hat{\Sigma})\}$, they are compact and convex.

1.4 Robust distortion risk measure model with Gelbrich Uncertainty Set

In this section, we establish the robust mean-distortion risk measure model with Gelbrich uncertainty set and introduce the simplified expression of the objective function.

Proposition 7 If $\rho(X)$is a law-invariant, positive homogenous risk measure, then the Gelibrich risk if the portfolio loss function $l(\xi) = -x^T\xi$is given by

$$\sup_{\mathcal{Q}\in\mathcal{G}_\pi(\hat{\mu},\hat{\Sigma})} \rho_{\mathcal{Q}}(-x^T\xi) = -\hat{\mu}^T x + \alpha\sqrt{x^T\hat{\Sigma}x} + \rho\sqrt{1+\alpha^2}\|x\|\#(8)$$

where

$$\alpha(\mu,\Sigma,x) = \sup_{\mathcal{Q}\in(\mathcal{C}(\mu,\Sigma))} \rho_{\mathcal{Q}}\left(-\frac{x^T(\xi-\mu)}{\sqrt{x^T\Sigma x}}\right)\#(9)$$

is called standard risk coefficient.

Proposition 8[20] If the distortion risk function g is a right continuous function, $\rho_Q$is the distortion risk measure based on the distortion risk function $\mathcal{Q} \in \mathcal{M}, \mathcal{S} = \mathcal{M}_2$, then the standard risk coefficient for the distortion risk measure is given by

$$\alpha = \left(\int_0^1 g'_{cvx}(\tau)^2 d\tau - 1\right)^{\frac{1}{2}}\#(10)$$

where g'_{cvx}denotes the derivative of the convex envelope of g, which exists almost everywhere. The proof of thisproposition can be found in the article of Li.

Therefore, when the distortion risk measure satisfies the properties of coherent and law-invariant, then $\sup_{\mathcal{Q}\in\mathcal{G}_\pi(\hat{\mu},\hat{\Sigma})} \rho_{\mathcal{Q}}(-x^T\xi)$ describes the worst-case distortion risk, the robust distribution uncertainty model based on Gelbrich uncertainty sets can be represented as

$$\min_{x\in\Omega} \sup_{\mathcal{Q}\in\mathcal{G}_\pi(\hat{\mu},\hat{\Sigma})} \rho_{\mathcal{Q}}(-x^T\xi) = \min_{x\in\Omega} -\hat{\mu}^T x + \alpha\sqrt{x^T\hat{\Sigma}x} + \pi\sqrt{1+\alpha^2}\|x\|$$

$$= \min_{x\in\Omega} -\hat{\mu}^T x + \left(\int_0^1 g'_{cvx}(\tau)^2 d\tau - 1\right)^{\frac{1}{2}}\sqrt{x^T\hat{\Sigma}x} + \pi\sqrt{\int_0^1 g'_{cvx}(\tau)^2 d\tau}\,\|x\|\#(11)$$

If the constraint of returnis added, this model can be written

$$\min_{x\in\Omega} \sup_{\mathcal{Q}\in\mathcal{G}_\pi(\hat{\mu},\hat{\Sigma})} \rho_{\mathcal{Q}}(-x^T\xi) =$$

$$\min_{x\in\Omega} -\hat{\mu}^T x + \left(\int_0^1 g'_{cvx}(\tau)^2 d\tau - 1\right)^{\frac{1}{2}}\sqrt{x^T\hat{\Sigma}x} + \pi\sqrt{\int_0^1 g'_{cvx}(\tau)^2 d\tau}\,\|x\|\#(12)$$

$$s.t.\ E_{\mathcal{Q}}[\xi] \geq r_0$$

According to the proposition raised by Viet [21](2019), any external distribution$\mathcal{Q}$that attains the Gelbrich risk of loss function $l(\xi) = -x^T\xi$has the same mean $\mu^*$and covariance matrix Σ^*.

Then the constraint of return is changed to this form

$$E_{\mathcal{Q}}[\xi] = x^T\hat{\mu} - \frac{\pi x^T x}{\sqrt{1+\alpha^2}\|x\|} = x^T\hat{\mu} - \frac{\pi x^T x}{\|x\|\int_0^1 g'_{cvx}(\tau)^2 d\tau} \geq r_0\#(13)$$

V. Robust model with safety criteria

1.5 Robust model with original safety criteria constraint

This section will refer to the safety criteria proposed by Li (2017) as constraints for robust optimization models. This constraint mainly indicates that investors assign different weights to the best and worst return rates, and comprehensively calculate their expected return rate under their risk preference, and require that the return meet certain goals. Similarly, if the objective function is a linear combination of the best- and worst-case scenarios, the final model will be as follows:

$$\min \ (1-\theta)\inf_{Q\in\mathcal{T}}\rho_Q(-x^T\xi)+\theta\sup_{Q\in\mathcal{T}}\rho_Q(-x^T\xi)$$
$$s.t.\ (1-\theta)\sup_{Q\in\mathcal{T}}E_Q\big(R(x,\xi)\big)+\theta\inf_{Q\in\mathcal{T}}E_Q\big(R(x,\xi)\big)\geq\phi \quad \#(14)$$

Among them, the return function $R(x,\xi)=x^T\xi$, $\phi$is the target for investors, $\theta\in[0,1]$, that characterizes the risk aversioncoefficient of an investor for uncertainty. When $\theta=0$,investors prefer more aggressive investment strategies, which only consider the maximum return in the best case. Once the maximum return exceeds the established target, this investment portfolio at least satisfies investors' risk preferences. Whenθapproaches 1, it indicates that investors are more conservative and pay more attention to the certainty and stability of returns, considering the achievement of return targets in worst-case scenarios.

Definition 4 Assuming a random vector of returns $\xi(\mu,\Sigma)\in R^n$ is derived from the following uncertain set

$$\mathcal{U}(\hat{\mu},\hat{\Sigma})=\left\{(\bar{\mu},\bar{\Sigma})\in R^n\times S^n: \begin{matrix}(\bar{\mu}-\hat{\mu})^T\hat{\Sigma}^{-1}(\bar{\mu}-\hat{\mu})\leq\gamma_1\\ \bar{\Sigma}\preccurlyeq\gamma_2\hat{\Sigma}\end{matrix}\right\} \#(15)$$

$\mathcal{U}(\hat{\mu},\hat{\Sigma})$ includes the information on the mean vector and covariance matrix, γ_1,$\gamma_2$represent the degree of the uncertainty. The higher figures of γ_1,γ_2, the higher level of ambiguity.

Proposition 9 Suppose a random vector of return $\xi(\mu,\Sigma)\in R^n$ is derived from the uncertain distribution set $\mathcal{T}$, and the robust optimization model under original safety criteria constraints can be expressed in the following form

$$\min_{x\in R^n} -\hat{\mu}^Tx+\left[(2\theta-1)\sqrt{\gamma_1}+\delta\theta\sqrt{\gamma_2}\right]\sqrt{x^T\hat{\Sigma}x}$$

$$s.t.\,\hat{\mu}^Tx+(1-2\theta)\sqrt{\gamma_1}\sqrt{x^T\hat{\Sigma}x}\geq\phi\#(16)$$

$\delta=\left(\int_0^1 g'_{cvx}(\tau)^2d\tau-1\right)^{\frac{1}{2}}$ denotes risk appetites of investors induced by different distortion risk functions is related to the parameters of the distortion risk function.

1.6 Robust model with probabilistic safety criteria constraint

In real world, investors do not strictly pursue a return level that meets their personal investment goals with a probability of 100%. Investors often exhibit a certain risk preference and overconfidence. They believe that a certain investment portfolio can exceed a predetermined return rate with a probability of more than 95%, which is acceptable. Therefore, this section will refer to Li's (2017) article and change the expected return constraint to a constraint under probability weighting, which introduces probability and parameters θ. The specific constraint formula for representing the risk preference of investors' investment strategies is

$$P\left\{(1-\theta)\sup_{Q\in\mathcal{T}}E_Q\big(R(x,\xi)\big)+\theta\inf_{Q\in\mathcal{T}}E_Q\big(R(x,\xi)\big)\leq\phi\right\}\leq\varepsilon\ \#(17)$$

$\phi$is the predetermined return target for investors, while the probability constraint $\varepsilon$is generally (0,0.5).

Proposition 10[21]The constraint of probabilistic safety criteria can be given by the following form

$$\frac{\gamma_2}{\varepsilon}x^T\hat{\Sigma}x-\sum_{k=1}^{n}\left\{(\hat{\mu}_k^Tx-\hat{\mu}_{k-1}^Tx)\left[\hat{\mu}_k^Tx+\hat{\mu}_{k-1}^Tx+2(1-2\theta)\sqrt{\gamma_1}\sqrt{x^T\hat{\Sigma}x}-2\phi\right]\right\}\leq 0$$

The process of the proof uses the Chebyshev's inequality and Bhat [22]'s result.

Proposition 11 Suppose a random vector of return $\xi(\mu,\Sigma)\in R^n$ is derived from the uncertain distribution set $\mathcal{T}$, the distortion risk measure robust optimization model based on theprobabilistic safety criterion can be transformed into the following form:

$$\min_{x\in R^n} -\hat{\mu}^Tx+\left[(2\theta-1)\sqrt{\gamma_1}+\delta\theta\sqrt{\gamma_2}\right]\sqrt{x^T\hat{\Sigma}x}$$

$$s.t. \frac{\gamma_2}{\varepsilon} x^T \hat{\Sigma} x - \sum_{k=1}^{n} \left\{ (\hat{\mu}_k^T x - \hat{\mu}_{k-1}^T x) \left[\hat{\mu}_k^T x + \hat{\mu}_{k-1}^T x + 2(1-2\theta)\sqrt{\gamma_1}\sqrt{x^T \hat{\Sigma} x} - 2\phi \right] \right\} \le 0$$

$\delta = \left(\int_0^1 g'_{cvx}(\tau)^2 d\tau - 1 \right)^{\frac{1}{2}}$ denotes risk appetites of investors induced by different distortion risk functions is related to the parameters of the distortion risk function.

VI. Empirical analysis

1.7 Empirical data

The sample data in this empirical analysis is mainly based on the Shanghai and Shenzhen stock markets, and 10 representative stocks from different industries, different listing times, and low yield correlation are selected from China's A-share market. They are Longji Machinery (002363), Yiling Pharmaceutical (002603), Suzhou Bank (002966), Compass (300803), Nongfa Seed Industry (600313), China People's Insurance Corporation (601319), Laiyifen (603777), Sanqi Mutual Entertainment (002555), Qingdao Beer (600600), and Jingsheng Electromechanical (300316). The sample data has the time range of 1 year, from January 1, 2022 to December 31, 2022, and the corresponding yield is calculated based on the opening and closing prices of each trading day. The following pictures mainly depict the closing price trends of the above 10 stocks. It can be seen that there are still differences in the trends of these stocks, which also verifies the low correlation between the selected stocks.

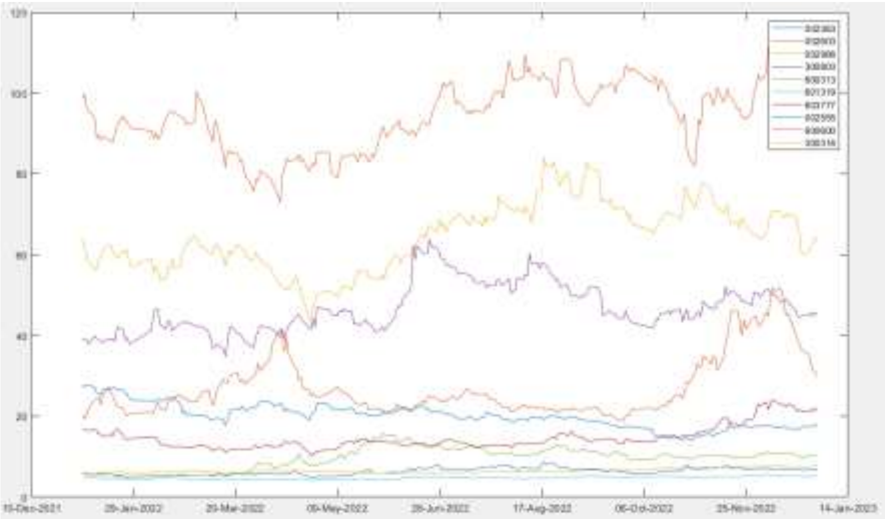

Figure 1 Closing price trend figures of 10 stocks

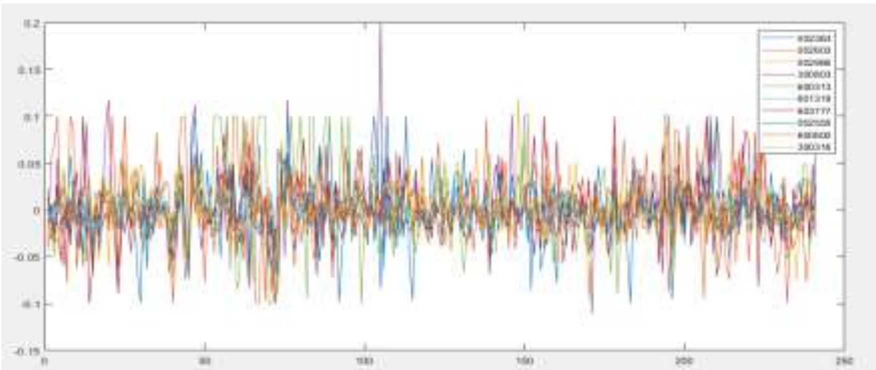

Figure 2 Daily return rate fluctuation figures of 10 stocks

Table 1 and Figure 3 depict the annual return, annual standard deviation, minimum, maximum, kurtosis, skewness, and correlation matrices of each stock, respectively.

Table 1 Statistics of 10 stocks

Code	N	Return	SD	min	max	kurtosis	skewness
002363	241	35.39%	0.558	-10.07%	10.08%	4.35	0.284
002603	241	65.40%	0.719	-10.00%	10.02%	2.93	0.318
002966	241	29.84%	0.239	-4.79%	5.21%	3.54	0.081
300803	241	30.89%	0.563	-11.03%	20.00%	7.39	1.187
600313	241	78.10%	0.657	-10.01%	10.07%	3.40	0.247
601319	241	16.33%	0.250	-6.34%	6.04%	5.24	0.322
603777	241	41.52%	0.539	-10.00%	10.01%	3.82	0.148
002555	241	-30.87%	0.436	-10.00%	10.03%	5.33	0.169
600600	241	17.23%	0.416	-8.75%	9.69%	4.99	0.608
300316	241	9.20%	0.457	-9.57%	11.96%	4.78	0.398

Figure 3 Correlation matrices of each stock

1.8 Empirical results under Gelbrich Uncertainty set

We take these two functions $g(x) = min\left(\frac{x}{1-\alpha}, 1\right)$and $g(x) = 1-(1-x)^{1+\lambda}$ with $\alpha = 0.95$as example.

The left figure in Figure 4 depicts the changes of effective frontier when π is taken as 0.1, 0.2and 0.3. The figure on the right shows the changes in the investment ratio of each asset under different Gelbrich distances. It can be observed that the smaller the value of Gelbrich distance, the more its effective frontier spreads outward. This property can be derived from the Gelbrich distance formula.

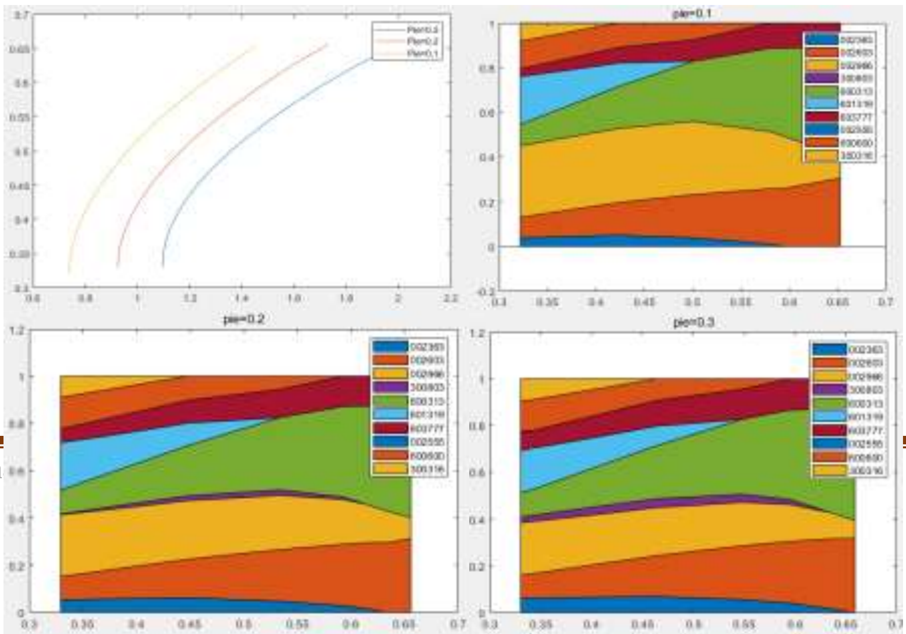

Figure 4 The results of different Gelbrich distances π

The parameter λ measures the loss probability of investors adjusting different positions under subjective conditions, indicating their risk preference. Therefore, the empirical results analyzed that under the fixed Gelbrich distanceλ. Please refer to Figure 5 for the specific changes in the effective frontier and investment ratio.

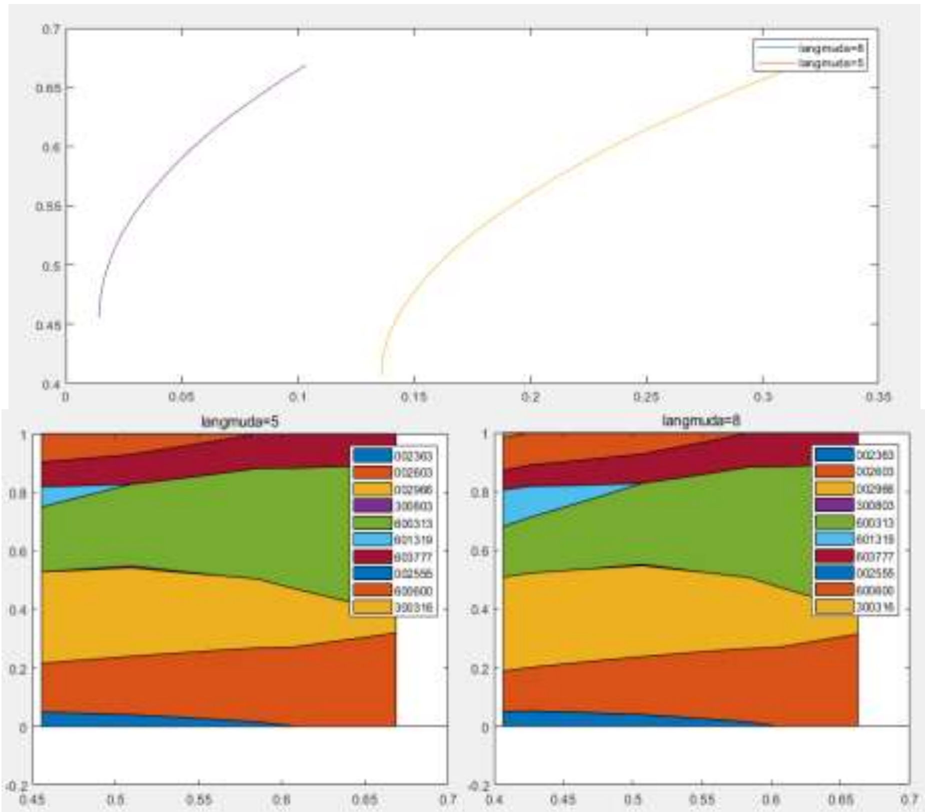

Figure 5 The results of different parameter λ

1.9 Empirical results under safety criteria

In this section, we use the bootstrapping method to estimate the parameters $\gamma_1$and γ_2. Also, to well show the change of the weight of each asset, we refer to the research of Li and consider only 3 stocks here. Through the bootstrapping procedure, we get $\hat{\gamma}_1 = 0.2140$and $\hat{\gamma}_2 = 0.1003$.

Whenα and λequal to 0.95 and 5, the results are shown in the following figure. Figure 6 shows the change of the effective frontier and investment proportion when $\alpha = 0.95$ and the aversion for uncertaintyθ with values of 0.5, 0.8and 1. It can be clearly seen from the figure that when θgets higher, the higher the expected return rate, and the smaller the value of the best and worst distortion risk measure. Investors have a stronger attitude towards avoiding uncertainty.

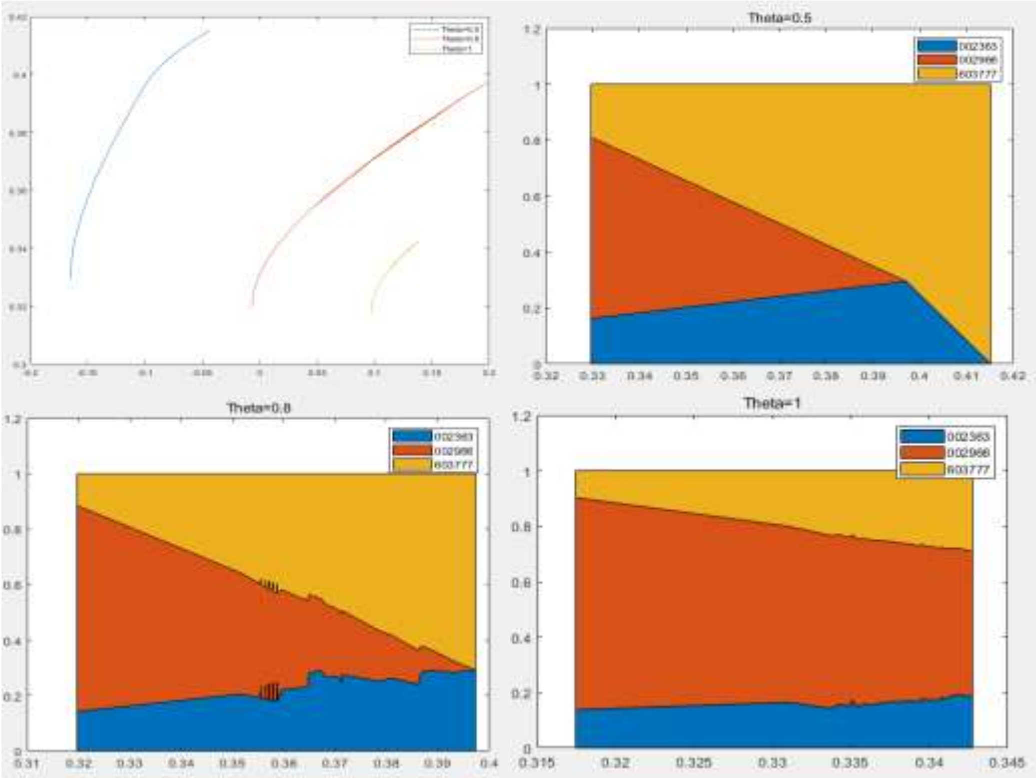

Figure 6 The results of different parameterθ

For the probabilistic safety criteria, the empirical results are shown in the figure 7.It can be observed that when θget larger, the more the effective frontier moves to the right, and the proportion of investment in stock 002966 also increases.

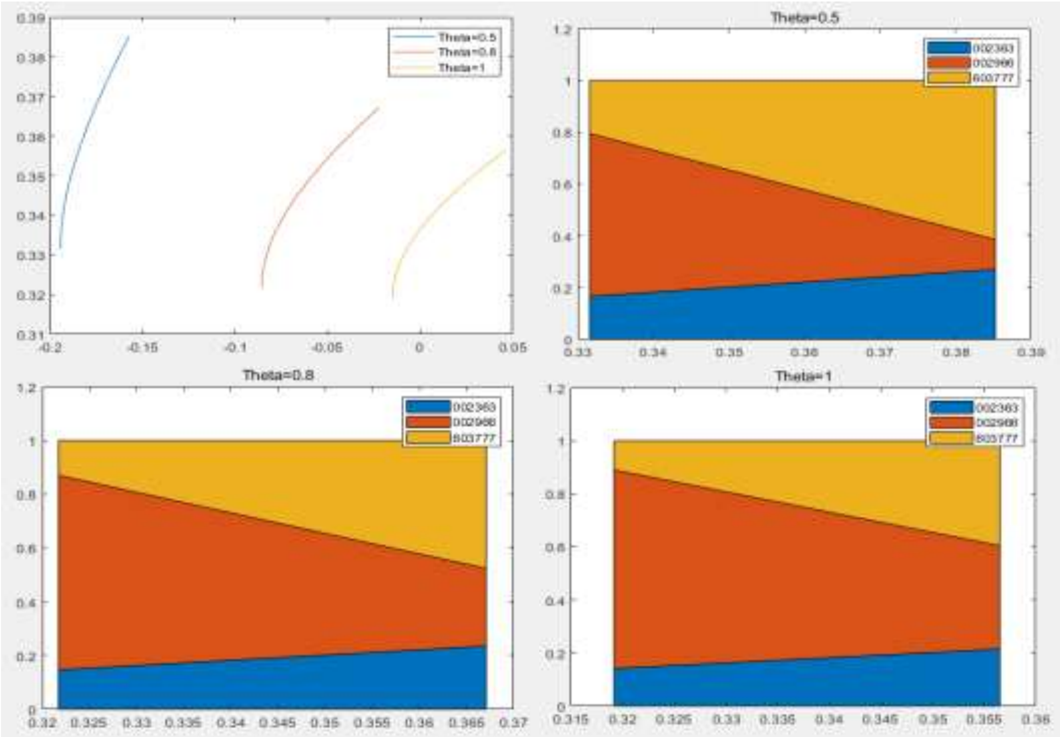

Figure 7 The results of different parameterθ

VII. Conclusion

The article studies the relevant properties of distortion risk measures based on distortion functions, and analyzes and solves the optimization model of mean distortion risk measures on this basis. Before model analysis, uncertain sets with different definitions were introduced to examine the changes in the effective frontier of the model under different uncertain sets. Further improve the constraint conditions of the model, introduce safety criteria constraints, and solve the changes in the optimal solution of the model under changes in relevant parameters. The results indicate that the effective frontier of the probability safety criterion shifts to the right relative to the original model, meaning that investors bear greater risk at the same rate of return. At the same time, the smaller the risk aversion coefficient of the uncertainty set, the more frontier moves to the left and up, which is consistent with the conclusion under general safety criteria.

References

[1] Markowitz H M. Portfolio selection[J]. The Journal of Finance, 1952, 7(1):77.

[2] Duffie D, Pan J. An overview of value at risk[J]. Journal of derivatives, 1997, 4(3): 7-49.

[3] Artzner P, Delbaen F, Eber J M, et al. Coherent Measures of Risk[J]. Mathematical Finance, 1999, 9(3):203-228.

[4] Delbaen,Freddy. Coherent risk measures on general probability spaces[J]. Advances in finance and stochastics: essays in honour of Dieter Sondermann, (2002): 1-37.

[5] Acerbi C. Spectral measures of risk: A coherent representation of subjective risk aversion[J]. Journal of Banking & Finance, 2002, 26(7):1505-1518.

[6] Wang S . Premium Calculation by Transforming the Layer Premium Density[J]. ASTIN Bulletin, 1996, 26.

[7] Wang S S . A Universal Framework for Pricing Financial and Insurance Risks[J]. Cambridge

University Press, 2002(02).

[8] A Balbás, Garrido J , Mayoral S . Properties of Distortion Risk Measures[J]. Springer US, 2009(3).

[9] Feng M B, Tan K S. Coherent Distortion Risk Measures in Portfolio Selection[J]. Systems Engineering Procedia, 2012, 4:25-34.

[10] Cai J , Wang Y , Mao T . Tail subadditivity of distortion risk measures and multivariate tail distortion risk measures[J]. Insurance Mathematics and Economics, 2017, 75(jul.):105-116.

[11] Gelbrich M. On a formula for the L2 Wasserstein metric between measures on Euclidean and Hilbert spaces[J]. MathematischeNachrichten, 1990, 147(1): 185-203.

[12] Delage, Erick, and Yinyu Ye. Distributionally Robust Optimization Under Moment Uncertainty with Application to Data-Driven Problems. Operations Research, 2010, 58(3): 595–612.

[13] Roy, A.D. (1952) Safety First and the Holding of Assets. Econometrica, 20, 431-449.

[14] TurnovskyP . The Dynamics of Government Policy in an Inflationary Economy" An `Intermediate-Run' Analysis. 1975.

[15] Kataoka, Shinji. A Stochastic Programming Model. Econometrica 31, no. 1/2 (1963): 181–96.

[16] Ghaoui L E, Oks M, Oustry F. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach[J]. Operations research, 2003, 51(4): 543-556.

[17] Zhu M S. Worst-Case Conditional Value-at-Risk with Application to Robust Portfolio Management[J]. Operations Research, 2009, 57(5):1155-1168.

[18] Kang Z, Li X, Li Z. Mean-CVaR portfolio selection model with ambiguity in distribution and attitude[J]. Journal of Industrial and Management Optimization, 2020, 16(6): 3065-3081.

[19] Nguyen, Viet Anh, et al. Mean-covariance robust risk measurement. arXiv preprint arXiv:2112.09959 (2021).

[20] Li J Y M. Closed-form solutions for worst-case law invariant risk measures with application to robust portfolio optimization[J]. Operations Research, 2018, 66(6): 1533-1541.

[21] Shihan Di, Dong Ma, Peibiao Zhao. α-robust portfolio optimization problem under the distribution uncertainty. Journal of Industrial and Management Optimization, 2023, 19(4): 2528-2548.

[22] Rujeerapaiboon N, Kuhn D, Wiesemann W. Chebyshev inequalities for products of random variables[J]. Mathematics of Operations Research, 2018, 43(3): 887-918.