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Research on the Optimal Portfolio Strategy under Logarithmic Utility with Uncertain Distribution

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Abstract : *In this present paper, we take the portfolio theory and the distribution uncertainty theory as the basis to consider the optimal portfolio problem under the uncertain distribution of risk assets. We establish the model combined with the maximin decision criterion (pessimistic criterion) and use the martingale method as well as the Hamilton-Jacobi-Bellman (HJB) equation to study and give the analytical solution of the model under the logarithmic utility function. At the last part, a numerical analysis is carried out and the result of it shows that the model constructed in this paper is reasonable and effective.*

Keywords - *Optimal portfolio, Distribution uncertainty, Utility maximization, Martingale method*

I. INTRODUCTION

In the research of modern portfolio theory, the problem of optimal portfolio strategy has always been the core proposition in the financial field. The optimal portfolio problem mainly studies how to allocate between risk-free assets and risky assets, so as to maximize the expected utility of investors' terminal wealth. Merton[1](1971) proposed the utility maximization problem for the first time, discussed the optimal consumption portfolio problem under the constant coefficient power utility function, and used the Markov continuous time random control method to solve the corresponding HJB equation, and finally Obtained an explicit solution to the optimal policy. Pliska[2](1986) introduced the martingale method to solve the optimal investment strategy problem for a more generalized price process in the complete market, and used the martingale representation theorem to describe the optimal investment strategy. In some specific cases, we can get an explicit solution to the problem. Among them, the constraints on the utility function are increasing and strictly concave. Zhao[4](2001) studied the portfolio optimization problem in the Bayesian case (ie, the drift rate is continuous but unknown) in the finite time horizon, by using the martingale method, local differential equations, and the HJB equation, he solved the utility maximization problem under the exponential utility function. Yuan, li[5](2010) used the martingale method to solve the optimal portfolio problem with a CRRA-type (constant relative risk aversion) utility function and the risk asset drift rate obeys a normal distribution, and solved the explicit solution when the learning was introduced.

In recent years, the research on distribution uncertainty mainly focuses on the construction of distribution uncertainty set, and reduces the disturbance to the optimal solution of the model by constraining the uncertain parameters. Fabozzi, Kolm[7](2009) constructed an ellipsoid distribution uncertain set, and established an ellipsoid uncertain set of first-order moments for the uncertain mean vector, using the parameter λ as the radius of the ellipsoid to describe Uncertain range for the mean vector. Kang, Kuhn[8](2018) used Wasserstein distance, a distance function in the probability space, to further expand the construction method of distribution uncertainty sets. Kang, Li[9](2019) used the first-order moments and second-order moments of the uncertain parameter mean and covariance to construct a distributional uncertainty set with conic constraints. And further more, he studied the mean-CVaR portfolio problem research On this basis.

This paper is mainly based on the portfolio theory and distribution uncertainty theory, studies the optimal portfolio problem under the unobservable drift rate of risky assets with uncertain distribution, and constructs the optimal portfolio model which is combined with the distribution uncertainty set and the maximin decision criterion (pessimistic criterion). At the same time, we solve the optimal portfolio problem, and give the expression of the optimal strategy solution under the logarithmic utility function, which has certain theoretical and practical significance.

The structure of this paper is as follows: In the second part, an optimal portfolio model which is combined with an uncertain distribution set is constructed. In the third part, the above optimal portfolio problem is solved, and the expression of the optimal strategy solution under the logarithmic utility function is given. The fourth part is numerical analysis, we simulate the optimal investment strategy with uncertain distribution under the logarithmic utility function, and explore the influence of each parameter on the value of the optimal strategy. The last part is summary and prospect, we summarize the theories and model involved in this paper, and give directions for further in-depth study of optimal portfolio problems based on distribution uncertainty and utility functions.

II. MODEL FORMULATION

Based on the optimal portfolio problem of li[5] (2010), this chapter introduces the distribution uncertainty set to construct a portfolio model in which the risk asset drift rate information cannot be observed by investors and the initial prior distribution is uncertain distribution.

Firstly, we consider the existence of two assets in the financial market: the first asset is a risk-free asset (bond) and the second asset is a risky asset (stock), and their price processes are subject to the following differential equations:

$$dB_t = rB_t dt, \quad 0 \le t \le T \#(2.1)$$

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad 0 \le t \le T \#(2.2)$$

Among them, $W = (W_t)$ is the standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F} = \mathcal{F}_t, 0 \le t \le T, P)$; The constant $r \ge 0$ is a risk-free interest rate, investors can accurately know the value of r, and can also accurately observe the price process of risk assets S_t; The constants σ and μ are the volatility and drift rate of the risky asset, respectively.

In the real market, the volatility and drift rate of risky assets cannot be accurately known by investors. However, there is a large amount of literature that believes that investors pay more attention to the drift rate of risky assets, and at the same time can withstand fluctuations in a large range. And there are also studies that show[11]: when predicting the expectation and variance of stock returns based on historical data on stock prices with a small sample size, the prediction of yield expectations is very inaccurate. Therefore, this paper considers the volatility σ of the risky asset to be a controllable known constant, while the drift rate μ is unknown and the distribution is uncertain, so the Brownian motion W_t is also unknowable.

Thus, the total wealth process at the t moment can be expressed as the following stochastic differential equation (SDE):

$$\begin{aligned} dX_t^\pi &= X_t^\pi \pi_t \frac{dS_t}{S_t} + X_t^\pi (1-\pi_t)\frac{dB_t}{B_t} \qquad \#(2.3) \\ &= X_t[(r + (\mu - r)\pi_t)dt + \sigma \pi_t dW_t] \end{aligned}$$

Where the strategy π_t represents the share of investors invested in stocks at the t moment, so $1 - \pi_t$ is the share invested in bonds at the t moment. When $\pi_t \le 0$, it means that the stock is sold short, and when $\pi_t \ge 1$, it means that the investor is in debt to buy the stock.

Next, we combine the conical distribution uncertainty set constructed by Kang, Li[9] (2019) using the mean and covariance of uncertain parameters to construct the first and second moment of the conical distribution, we optimize and improve this set, and apply it to the setting of risk asset drift rate μ. We consider that the prior distribution of the risk asset drift rate μ at the beginning of the period has the following distribution uncertainty set $\mathbb{D}_F(\gamma_1)$:

$$\mathbb{D}_F(\gamma_1) = \left\{ \begin{array}{c} P(\mu \in \Omega) = 1,\ P \in \mathcal{M}_+ \\ \left(E_p(\mu) - \hat{\mu}\right)^T \sigma^{-1} \left(E_p(\mu) - \hat{\mu}\right) \le \gamma_1 \end{array} \right\} \#(2.4)$$

Among them, $\Omega = \mathbb{R}^n$ is the sample space. $E_p(\mu)$ represents the expected mean of investors for the drift rate μ under the uncertain distribution P; $\hat{\mu} \in R^n$ represents the mean vector of the sample drift rate; The magnitude of the uncertainty level parameters γ_1 measures the magnitude of the uncertainty of the drift rate's expected mean.

Next, let's assume that the random variable $\mu \in R^n$ has a priori distribution at the beginning of the period, and its mean $\overline{\mu}$ comes from the set $F(\overline{\mu})$, there is

$$F(\overline{\mu}) = \{P \in \mathcal{M}_+ : E_p(\mu) = \overline{\mu}\}\#(2.5)$$

At the same time, the $\overline{\mu}$ also belongs to the set $\mathcal{U}$, ie

$$\mathcal{U}(\hat{\mu}) = \left\{ \begin{array}{c} P(\overline{\mu} \in \Omega) = 1, \overline{\mu} \in R^n \\ (\overline{\mu} - \hat{\mu})^T \sigma^{-1} (\overline{\mu} - \hat{\mu}) \le \gamma_1, \end{array} \right\}\#(2.6)$$

Therefore, there is

$$\mathbb{D}_F(\gamma_1) = \bigcup_{\overline{\mu} \in \mathcal{U}(\hat{\mu})} F(\overline{\mu})\#(2.7)$$

When the prior distribution of the random variable $\mu \in R^n$ satisfies the distribution uncertainty set $F(\overline{\mu})$ at the beginning of the period, the total wealth process at the t moment satisfies:

$$\begin{cases} X_t^\pi = x_0 \exp\left\{ \int_0^t \left(r + \left(E_p[\mu|\mathcal{F}_t^S] - r \right) \pi_s - \frac{1}{2}\sigma^2 \pi_s^2 \right) ds + \int_0^t \sigma \pi_s dW_s \right\} \\ E_p[\mu|\mathcal{F}_0^S] = \overline{\mu} \end{cases} \#(2.8)$$

wherein $E_p[\mu|\mathcal{F}_t^S]$ can be understood as the investor"s "learning": at the t moment, under the uncertain distribution P, the optimal estimate of the expected drift rate μ based on the observed existing stock price information; $E_p[\mu|\mathcal{F}_0^S]$ represents the average investor's expectation of the stock price drift rate μ at the beginning of the period, that is, equal to $\overline{\mu}$, when $\overline{\mu}$ is larger, it indicates that investors have more optimistic investment beliefs.

Lemma 1 (Chen et al.[19] 2011). *Define the uncertainty set* $\mathcal{U}_{\hat{\mu}} = \{\overline{\mu} \in R^n : (\overline{\mu} - \hat{\mu})^T \sigma^{-1} (\overline{\mu} - \hat{\mu}) \le \gamma_1\}$ *with respect to the* $\overline{\mu}$. *The lower limit of the* $\overline{\mu}$ *can be expressed as:*

$$\inf_{\overline{\mu} \in \mathcal{U}_{\hat{\mu}}} \overline{\mu} = \hat{\mu} - \sqrt{\gamma_1 \sigma}\#(2.9)$$

According to maximin decision criterion (pessimistic criterion), pessimistic decision-makers tend to believe that there will be the worst investment environment in the future, and in order to minimize risks, decision-makers will pay attention to the benefits of each solution in the worst-case scenario, and choose the most profitable solution as the optimal investment solution. Therefore, based on this criterion, we consider the issue of maximizing the expected utility of end-term wealth in the worst-case scenario of the expected return of the stock price:

$$\max_{\pi_t \in \mathbb{R}} E\left[U(X_T^\pi)|\mathcal{F}_0^S\right]$$

$$s.t. \begin{cases} dX_t^\pi = X_t\left[\left(r + (E_p[\mu|\mathcal{F}_t^S] - r)\pi_t\right)dt + \sigma \pi_t dW_t\right] \\ E_p(\mu|\mathcal{F}_0^S) = \inf_{P \in \mathbb{D}_F} E_p(\mu|\mathcal{F}_0^S) \\ U(x) = lnx, \quad x > 0 \\ X_0 = x_0 \end{cases} \#(2.10)$$

Among them, the size of the expected mean of the stock price drift rate μ at the beginning of the period $E_p[\mu|\mathcal{F}_0^S]$ is the value of its inferior definum in the uncertain distribution set $\mathbb{D}_F$.

Therefore, combining the definition of the distribution uncertainty set $\mathbb{D}_F(\gamma_1)$ and $\mathcal{U}(\hat{\mu})$ and lemma 2.1, we can translate the above optimal portfolio problem of the return distribution of risky assets into an uncertain situation:

$$\max_{\pi_t \in \mathbb{R}} E\left[U(X_T^\pi)|\mathcal{F}_0^S\right]$$

$$s.t. \begin{cases} dX_t^\pi = X_t\left[\left(r + (E_p[\mu|\mathcal{F}_t^S] - r)\pi_t\right)dt + \sigma \pi_t dW_t\right] \\ E_p(\mu|\mathcal{F}_0^S) = \hat{\mu} - \sqrt{\gamma_1 \sigma} \\ U(x) = lnx, \quad x > 0 \\ X_0 = x_0 \end{cases} \#(2.11)$$

Among them, the utility function $U(x)$ is a strictly concave, strictly monotonically increasing quadratic continuous differentiable function, and $\mathcal{F}_0^S$ is the information that investors have for the assets they hold at 0 (at the beginning of the period). At the same time, the stock price drift rate μ cannot be observed by investors, but investors know that they have an expectation of $\hat{\mu} - \sqrt{\gamma_1 \sigma}$.

III. SOLUTION TO THE PORTFOLIO PROBLEM

In this chapter, we use the martingale method of Lakner[10] (1998), Girsanov's theorem, Bayes rule, etc., to solve the expression of the optimal investment strategy with uncertain distribution under the logarithmic utility function when the investors' expectation for the return on risk assets is the worst.

Theorem 3.1 (Girsanov's theorem[13]). *$B_t, 0 \le t \le T$ is a Brownian motion defined on the probability space $(\Omega, \mathcal{F}_t, P)$. the $\theta_t, 0 \le t \le T$ is a $\mathcal{F}_t$ fit process. defining $\tilde{B}_t = \int_0^t \theta_u \, du + B_t$, $Z_t = \exp - \int_0^t \theta_u \, dB_u - \frac{1}{2}\int_0^t \theta_u^2 \, du$, while defining a new probability measure $\tilde{P}(A) = \int_A Z_T \, dP$. Thus, under $\tilde{P}$ the procedure $\tilde{B}_t, 0 \le t \le T$ is a Brownian motion.*

Theorem 3.2.[30] *The procedure $\frac{1}{\zeta}$ is about the martingale under the probability measure $\tilde{P}$ and satisfies the following stochastic differential equation:*

$$d\left(\frac{1}{\zeta_t}\right) = \frac{1}{\zeta_t}\frac{(\mu_t - r)}{\sigma} dY_t \#(3.1)$$

Using the Ito formula, the explicit expression for ζ_t is:

$$\zeta_t = \exp\left\{-\int_0^t \frac{(\mu_u - r)}{\sigma} dY_t + \frac{1}{2}\int_0^t \frac{(\mu_u - r)^2}{\sigma^2} du\right\} \#(3.2)$$

Theorem 3.3.[30] *Under the risk-neutral measure $\tilde{P}$, the discount process $e^{-rt}X_t$ of a stock is martingale and satisfies:*

$$\tilde{E}[e^{-rT}X_T|\mathcal{F}_t^S] = E[e^{-rT}\zeta_T X_T|\mathcal{F}_t^S] \#(3.3)$$

Lemma 3.1 (Steven E. Shreve 2004[13]). *$Z_t = \exp - \int_0^t \theta_u \, dB_u - \frac{1}{2}\int_0^t \theta_u^2 \, du, 0 \le t \le T$ is the martingale under the probability measure P.*

Lemma 3.2 (Bayes rule[4]). *If X is $\mathcal{F}_t$ measurable, then for $0 \le s \le t \le T$, there is $\tilde{E}[X|\mathcal{F}_s] = \frac{1}{Z(s)} E[XZ(t)|\mathcal{F}_s]$.*

For the above optimal portfolio problem (Equation (2.11)) where the stock price drift rate is unknown and is under uncertain distribution, we define a new measure of risk neutrality $\tilde{P}$, and a new Brownian motion Y under $\tilde{P}$ based on theorem 3.1 above and lemma 3.1 and lemma 3.2:

$$\tilde{P}(A) = \int_A Z_T \, dP \#(3.4)$$

$$Y_t = W_t + \frac{(\mu_t - r)}{\sigma} t \#(3.5)$$

among them,

$$Z_t = \exp\left\{-\frac{(\mu - r)}{\sigma} W_t - \frac{(\mu - r)^2}{2\sigma^2} t\right\} \#(3.6)$$

Under the probability measure $\tilde{P}$, the total wealth process X_t^π satisfies the following stochastic differential equation:

$$dX_t^\pi = X_t[rdt + \sigma\pi_t dY_t] \#(3.7)$$

Using the Ito formula, we can know the process:

$$\begin{aligned} d(e^{-rt}X_t) &= -re^{-rt}X_t dt + e^{-rt}dX_t \\ &= e^{-rt}\sigma\pi_t dY_t \end{aligned} \#(3.8)$$

According to theorems 3.2 and 3.3, we transform the above dynamic portfolio model with unobservable terms into a fully measurable static model (Equation 3.9):

$$\begin{aligned} &\max_{\pi_t \in \mathbb{R}} E\,[U(X_T)|\mathcal{F}_0^S] \\ &s.t. \begin{cases} E[e^{-rT}\zeta_T X_T|\mathcal{F}_0^S] = x_0 \\ U(x) = \ln x \end{cases} \end{aligned} \#(3.9)$$

To solve the above static problem, we define $\eta_t = e^{-rt}\zeta_t$. Thus, the state price density process η_t can be expressed as:

$$\eta_t = \exp\left\{-rt - \int_0^t \theta_u \, dY_u + \frac{1}{2}\int_0^t \theta_u^2 \, du\right\}, \quad \eta_0 = 1 \#(3.10)$$

With the Ito formula, the explicit expression of the η_t is:

$$\eta_t = \exp\left\{-rt - \frac{1}{2}\gamma_0^t\theta_t^2 + \frac{1}{2}ln(\gamma_0^{t+1}) + \frac{1}{2}\frac{\left(E_p(\mu|\mathcal{F}_0^S) - r\right)^2}{\sigma})\right\}\#(3.11)$$

Next, we construct the Lagrange function:

$$\begin{cases} L(x,\lambda) = E[\ln X_T|\mathcal{F}_0^S] - \lambda\{E[\eta_T X_T|\mathcal{F}_0^S] - x_0\} \\ L_x = E[X_T^{-1}|\mathcal{F}_0^S] - \lambda E[\eta_T|\mathcal{F}_0^S] = 0 \\ L_\lambda = E[\eta_T X_T|\mathcal{F}_0^S] - x_0 = 0 \end{cases}\#(3.12)$$

We can get that

$$E[X_t^{-1}|\mathcal{F}_0^S] = \lambda E[\eta_T|\mathcal{F}_0^S]\#(3.13)$$

For the above equation, we take the expectations for the left and right sides of it, and can get the optimal terminal wealth X_T^* is:

$$X_T^* = \lambda\eta_T\#(3.14)$$

According to the martingale representation theorem, the optimal wealth process X_t^* can be expressed as:

$$\begin{aligned} X_t^* &= e^{-r(T-t)}E\left[\frac{\eta_T}{\eta_t}X_T^*|\mathcal{F}_t^S\right] \\ &= \frac{e^{-r(T-t)}\lambda}{\eta_t}E[\eta_T^2|\mathcal{F}_t^S] \end{aligned}\#(3.15)$$

According to the Ito formula, the coefficient before dY_t of the random term in dX_t^* is:

$$\frac{\partial X_t^*}{\partial\theta_t}\gamma_t\#(3.16)$$

Comparing the coefficients before dY_t in Equation (3.8), we obtain the expression for the optimal investment strategy π^* under the logarithmic utility function:

$$\begin{aligned} \pi^* &= \frac{E_p(\mu|\mathcal{F}_0^S) - r}{\sigma^2} \\ &= \frac{(\hat{\mu} - \sqrt{\gamma_1\sigma}) - r}{\sigma^2} \end{aligned}\#(3.17)$$

IV. NUMERICAL ANALYSIS

In this chapter, we perform a numerical simulation of the expression of the optimal investment strategy with uncertain distribution under the logarithmic utility function (Equation 3.17). where $\hat{\mu} \in R^n$ represents the mean vector of the sample drift rate, which satisfies:

$$\hat{\mu} = \frac{1}{S}\sum_{i=1}^{S}\mu_i\#(4.1)$$

We assume that the real volatility of the risky asset $\sigma = 4$, the risk-free rate of return $r = 0.03$, and the parameter $\gamma_1 = 0.02$, which measures the uncertain level of distribution. When the mean of the sample drift rate is $\hat{\mu} = 0.08$, we can find the value of the optimal investment strategy π^* as:

$$\begin{aligned} \pi^* &= \frac{(\hat{\mu} - \sqrt{\gamma_1\sigma}) - r}{\sigma^2} \\ &= \frac{\left(0.08 - \sqrt{0.02 * 4}\right) - 0.03}{4^2} \\ &= -0.015 \end{aligned}\#(4.2)$$

For different samples, we can get the function graph of the optimal investment strategy π^* with respect to the mean $\hat{\mu}$ as shown in Figure 1 below:

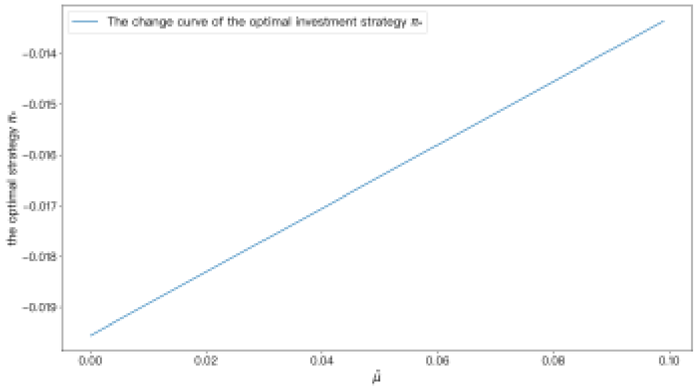

Figure 1. The effect of parameter $\hat{\mu}$ on the optimal strategy π^*

As can be seen from Figure 1, the mean value $\hat{\mu}$ has a positive effect on the value of the optimal investment strategy π^*. The size of the optimal strategy π^* increases as the parameter $\hat{\mu}$ increases.

In the distribution uncertainty set constructed in this paper, the distribution uncertainty parameter γ_1 quantifies the uncertainty of the expected mean of the stock price drift rate. When the value of the uncertainty parameter is larger, the degree of uncertainty is higher. Next, we let the real volatility of the risk asset $\sigma = 4$, the risk-free return $r = 0.03$, the mean of the sample drift rate $\hat{\mu} = 0.08$. At this time, we can get the function plot of the parameter γ_1 of the optimal investment strategy π^* about the uncertainty level of the distribution as shown in Figure 2 below:

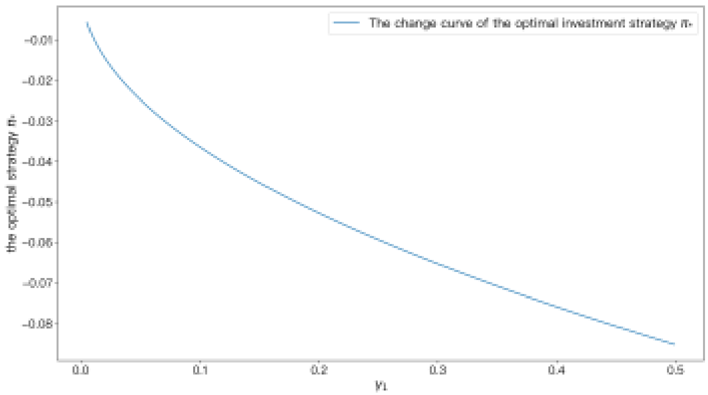

Figure 2. The effect of parameter γ_1 on the optimal strategy π^*

As can be seen from Fig.2, the value of the optimal investment strategy π^* changes in the opposite direction to the distribution uncertainty level parameter γ_1. As the parameter γ_1 increases, the optimal strategy π^* decreases, and the degree of reduction gradually decreases. This shows that as the uncertainty level of the return on risk assets increases, investors are more inclined to reduce their holdings of risk assets, and the degree of reduction gradually decreases with the increase of uncertainty level, which is consistent with the reality.

Through the above numerical analysis, we can conclude that the optimal portfolio model based on uncertain distribution constructed in this paper is reasonable and effective, and has good applicability.

V. SUMMARY AND PROSPECT

In the portfolio theory, investors make each investment decision based on the probability distribution of the return rate of risk assets, but in the real market, the return distribution of risk assets is affected by a variety of factors, and investors often cannot use limited information to accurately know the probability distribution of the return rate of risk assets. It is based on these practical factors that we introduce uncertain distribution to the portfolio, establish the optimal portfolio model under uncertain distribution of risk asset drift rate, and solve the optimal solution under the logarithmic utility function. In building our model, we consider the maximization of the expected utility of investors for end-of-life wealth in the worst-case scenario of expected returns on stock prices which is based on the pessimistic criteria. Finally, we conduct numerical analysis on the analytical solution of the model constructed in this paper, and the numerical analysis results show that the optimal portfolio model based on uncertain distribution constructed in this paper is reasonable and effective.

Due to the limitations of my own professional level, the problem studied in this paper still have shortcomings and need to be improved: for example, in the construction of the distribution uncertainty set, a higher-order and finer uncertainty set can be introduced; This paper only considers the portfolio model composed of one risky asset and one risk-free asset, and the types and number of assets are small, which can be enriched; In addition, more complex utility functions can be considered. For the above deficiencies, it is hoped that future scholars can do further research.

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