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Safety-first model with investor's view under distribution ambiguity

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Abstract: *In this paper, the black-litterman model is introduced to quantify investor's views, then we expanded the safety-first portfolio model under the case that the distribution of risk assets return is ambiguous. When short-selling of risk-free assets is allowed, the model is transformed into a second-order cone optimization problem with investor views. The ambiguity set parameters are calibrated through programming, then we use the interior point method to calculate the sensitivity of the optimal solution and the effective frontier. This paper finds that the overall effective frontier of the safety-first portfolio under distribution ambiguity is located above the original effective frontier, which means the safety-first portfolio with ambiguous distribution has stronger robustness.*
Keywords: *investor's views, Black-Litterman* model, safety-first rule, ambiguity set, Robust optimization.

I. Introduction

The Black-Littleman model was put forward by Fischer Black and Robert Litterman in 1992, which was based on the CAPM model, tried to incorporate investor's views based on a prior distribution to derive a posterior distribution of portfolio returns and optimal asset allocations [1]. Meucci [2] [3] [4] rephrased the model in terms of investors' views on the market and the market-based version was believed to be much more parsimonious and allowed for a more natural extension to directly input views in a non-Normal market, rather than just the market parameters as in the original Black and Litterman model. Then, Meucci extended the Black-Litterman model by the non-normal views, and allow for both analysis of the full distribution as well as scenario analysis. However, the implementation of Meucci's framework proposed so far still relies on restrictive normal assumptions.

Xiao and Valdez [5] made important work to further extend Meucci's model from the normal distribution to the elliptic distribution when market returns follow the ellipse distribution, however the assumption still based on market equilibrium in the model. After considering the appropriate conditional conjugate prior distribution, the explicit form of the posterior distribution was derived, so that the generalized model could be applied to various risk measures (such as mean variance, mean-VaR, mean-CVaR). On the basis of this work, Pang and Karan [6] also obtained the analytical solution of the optimal portfolio under the constraint of the mean-CVaR and the Black-Litterman model when the asset return rate follows an elliptic distribution.

In order to control the probability of loss and obtain the maximum safety returns, Kataoka [7] proposed the third form of the Kataoka-safety first rule (KSF), which was based on safety first rule and pursued the improvement of portfolio return under the constraint of the safety return. Elton and Gruber [8], introduced the KSF model under the assumption of normality; Ding and Zhang [9] conducted a further study on the KSF model with a normal distribution of asset returns under no short selling restrictions. They considered the geometric characteristics of the KSF model and established a risk asset pricing model to give an explicit solution to the optimal portfolio when selling short is allowed. Ding and Liu [10] also studied the problem of the optimal solution of the KSF model with risk-free assets, and found the optimal solution of the primary safety portfolio, and proved that this optimal solution was also effective for mean-variance .

However, Ellsberg [11] found many factors which were beyond the knowledge of most investors cause ambiguity or Knightian uncertainty in reality. Researches on the ambiguity focus on the distributionally robust formulation of the risk measures which described the worst-possible risk level over a set of potential distributions. The optimization is to be done subject to an ambiguity set of distributions rather than assuming that there is an underlying probability distribution that is known to the decision maker when it comes to distributionally robust optimization. It is a generalization of classical robust optimization which is well-known that the classical robust optimization can generate portfolios that are immune to noise and uncertainty in the parameters. But, Cheng et al. [12] found that such portfolios can be overly conservative because of making no use of any available information on distribution.For the reason that assuming probability distributions of uncertain parameters belong to an ambiguity set is the key ingredient of any distributionally robust optimization model, two types of ambiguity sets have been proposed: moment-based ambiguity sets and metric-based ambiguity sets. Popescu [13]; Delage [14] and Kang [15] studied moment-based ambiguity sets by assuming that all distributions in the distribution family satisfy certain moment constraints.Metric-based ambiguity sets, such as the Φ-divergence proposed by Bayraksan and Love [16] and Wasserstein metric studied by both Esfahani [17] and Jiang [18], may contain all distributions that were most likely distribution with respect to a probability metric or sufficiently close to a reference distribution.

II. The model

Consider the problem of a KSF (Kataoka-safety first rule) investor operating in a market consisting of n risky assets and a risk-less asset with return rate. Return of the i th risky asset follows the ellipse distribution, $r_i\ ED_n\left(\mu, D, g_n\right)$. Mean vector of risky assets is represented as $\mu = \mathrm{E}[r] \in \mathrm{R}^n$. $\Sigma \in \mathrm{R}^{n\times n}$ is the the covariance matrix of risk assets and $\Sigma \equiv cov(r) = \left(cov\left(r_i, r_j\right)\right)_{n\times n}$. μ is finite and $\Sigma \in \mathrm{R}^{n\times n}$ is finite and positive semidefinite. Strategy $\mathrm{x} \in \mathrm{R}^n$, $\mathrm{x}=\left(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_n\right)^T$. The variance of the risky assets portfolio can be represented as $var(r^T\mathrm{x}) \equiv \sigma_{\mathrm{x}}^2 = \mathrm{x}^T\Sigma\mathrm{x}$.

Lemma 2.1 $[19]$ *The KSF portfolio problem:*

$$\max_{\mathrm{x}} \quad r_{SF-\alpha}$$
$$\text{s.t.} \quad \begin{cases} P\left(\mathrm{E}\left[\tilde{v}\right] < r_{SF-\alpha}\right) \le \alpha \\ \mathrm{E}\left[\tilde{v}\right] = \mu^T\mathrm{x} + \left(1 - e^T\mathrm{x}\right) r_f \ge m \\ \mathrm{x}^T e - 1 > 0 \\ \mathrm{x}_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{2.1}$$

is equivalent to this optimization problem:

$$\max_{\mathrm{x}} \quad \mu^T\mathrm{x} + \left(1 - e^T\mathrm{x}\right) r_f + ED_\alpha\sqrt{\mathrm{x}^T\Sigma\mathrm{x}}$$
$$\text{s.t.} \quad \begin{cases} \left(1 - \mathrm{x}^T e\right) r_f + \mathrm{x}^T\mu \ge m \\ \mathrm{x}^T e - 1 > 0 \\ \mathrm{x}_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{2.2}$$

ED_α is a parameter decided by both confidence level α and the distribution of risky assets. Then, we can use the Black-Littleman model to quantify investor views. View matrix v is a $k\times n$ matrix, and its prior distribution is $v\,|\,r\ ED_k\left(\mathrm{P}\mu, \Omega, g_k(\cdot; p(r))\right)$, P is the view-choosing matrix and Π is Equilibrium market return matrix, $p(r) = (r - \Pi)^T \mathrm{D}^{-1}(r - \Pi)$.

Lemma 2.2 $[5]$ $r\,|\,\mathbf{v}\ ED_k\left(\mu_{BL}, \Sigma_{BL}, g_n(\cdot; q(\mathbf{v}))\right)$ *is the conditional distribution of risky assets . The parameters are calculated as follows:*

$$\mu_{BL} = \Pi + D\mathrm{P}^T\left(\Omega + \mathrm{P}\Sigma\mathrm{P}^T\right)^{-1}(v - \mathrm{P}\Pi) \tag{2.3}$$

$$D_{BL} = \Sigma - D\mathrm{P}^T\left(\Omega + \mathrm{P}D\mathrm{P}^T\right)^{-1}\mathrm{P}D \tag{2.4}$$

$$q(v) = (v - \mathrm{P}\Pi)^T\left(\Omega + \mathrm{P}\Sigma\mathrm{P}^T\right)^{-1}(v - \mathrm{P}\Pi) \tag{2.5}$$

$$\Sigma_{BL} = D_{BL}C_k(q(v)/2) \tag{2.6}$$

The function $C_k : R^+ \to R^+$ satisfies

$$C_k(x/2)g_k(x) = \frac{1}{2}\int_x^\infty g_k(t)dt \tag{2.7}$$

By using Black-Littleman model, we can replace μ and Σ in the original KSF problem with μ_{BL} and Σ_{BL} respectively, which are new parameters with investor's views. So, processed by Black-Littleman model, the BL- KSF problem can be written as follows:

$$\begin{aligned} \max_{\mathrm{x}} \quad & \mu_{BL}{}^T \mathrm{x} + \left(1 - e^T \mathrm{x}\right) r_f + ED_\alpha \sqrt{\mathrm{x}^T \Sigma_{BL} \mathrm{x}} \\ \text{s.t.} \quad & \begin{cases} \left(1 - \mathrm{x}^T e\right) r_f + \mathrm{x}^T \mu_{BL} \geq m \\ \mathrm{x}^T e - 1 > 0 \\ \mathrm{x}_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \end{aligned} \tag{2.8}$$

Definition 2.1 *(Ambiguity in distribution). The random variable μ_{BL} and Σ_{BL} assumes a distribution from the following set:*

$$\mathrm{D}_{BL}(\gamma_1, \gamma_2) = \left\{ \mathrm{P} \in \mathrm{M}_+ : \begin{array}{c} \left(\mathrm{E_P}(r_{BL}) - \hat{\mu}_{BL}\right)^{\mathrm{T}} \hat{\Sigma}_{BL}^{-1} \left(\mathrm{E_P}(r_{BL}) - \hat{\mu}_{BL}\right) \leq \gamma_1 m \\ \mathrm{Cov_P}(r_{BL}) \preceq \gamma_2 \hat{\Sigma}_{BL} \ , \ \mathrm{Cov_P}(r_{BL}) \succ 0 \end{array} \right\}$$

where M_+ is the set of all probability measures on the measurable space $\left(\mathrm{R}^n, \mathrm{B}\right)$ with the σ-algebra B on R^n. $\gamma_1, \gamma_2 \in \mathrm{R}^+$ are two scale parameters controlling the size of the uncertainty set. $\hat{\mu}_{BL}$ and $\hat{\Sigma}_{BL}$ are estimated values of the mean vector and the covariance matrix of μ_{BL} and Σ_{BL}. Here, $\preceq$ ($\prec$)means that if $A \preceq B$, then $B - A$ is positive semidefinite(positive definite).

With this Ambiguity set, the optimal strategy of BL-KSF model can be expressed explicitly and makes it more convenient for the later analysis on the behavior of investors with different levels of ambiguity aversion. We define the following two sets, which will be used in subsequent proofs. In $\mathrm{F}(\bar{\mu}_{BL}, \bar{\Sigma}_{BL})$, risky assets return r_{BL} satisfies:

$$\mathrm{F}(\bar{\mu}_{BL}, \bar{\Sigma}_{BL}) = \left\{ \mathrm{P} \in \mathrm{M}_+ : \mathrm{E_P}(r_{BL}) = \bar{\mu}_{BL}, Cov_{\mathrm{P}}(r_{BL}) = \bar{\Sigma}_{BL} \succ 0 \right\} \tag{2.9}$$

Its mean return $\bar{\mu}_{BL}$ and covariance matrix $\bar{\Sigma}_{BL}$ are in the Ambiguity set $\mathrm{U}(\hat{\mu}_{BL}, \hat{\Sigma}_{BL})$,

$$\mathrm{U}(\hat{\mu}_{BL}, \hat{\Sigma}_{BL}) = \left\{ (\bar{\mu}_{BL}, \bar{\Sigma}_{BL}) \in \mathbb{R}^n \times \mathrm{S}^n : \begin{array}{c} (\bar{\mu}_{BL} - \hat{\mu}_{BL})^{\mathrm{T}} \hat{\Sigma}_{BL}^{-1} (\bar{\mu}_{BL} - \hat{\mu}_{BL}) \leq \gamma_1 m \\ \bar{\Sigma}_{BL} \preceq \gamma_2 \hat{\Sigma}_{BL} \end{array} \right\}$$

where S^n denotes the space of the symmetric matrices of dimension n. Apparently,

$$\mathrm{D}_{BL}(\gamma_1, \gamma_2) = \bigcup_{(\bar{\mu}_{BL}, \bar{\Sigma}_{BL}) \in \mathrm{U}(\hat{\mu}_{BL}, \hat{\Sigma}_{BL})} \mathrm{F}(\bar{\mu}_{BL}, \bar{\Sigma}_{BL})$$

Based on the historical data of the return on risk assets-r_{BL}, we can calculate $\hat{\mu}_{BL}$ and $\hat{\Sigma}_{BL}$, and assume that $\hat{\mu}_{BL}$ and $\hat{\Sigma}_{BL}$ are unbiased estimates of $\bar{\mu}_{BL}$ and $\bar{\Sigma}_{BL}$. However, the deviation of estimates make the calculated portfolio do not have robustness in the worst case. In order to calculate the optimal asset portfolio of the robust BL-KSF model, we assume that the uncertainty set $\mathrm{U}(\hat{\mu}_{BL}, \hat{\Sigma}_{BL})$ accounts for information about the mean of random terms $\bar{\mu}_{BL}$ and the covariance matrix $\bar{\Sigma}_{BL}$. The robust model of BL-KSF can be expressed as follows:

$$\max_{x} \min_{P\in D_{BL}(\gamma_1,\gamma_2)} E\left(r_{BL}^T x + \left(1 - e^T x\right) r_f\right)$$
$$\text{s.t.} \max_{P\in D_{BL}(\gamma_1,\gamma_2)} P\left((r_{BL}^T - r_f e^T)x < m - r_f\right) \le \alpha, x \in X \tag{2.10}$$

In which,

$$X = \{x : x^T e - 1 > 0, x_i \ge 0, \quad i = 1, 2, \cdots, n\}.$$

III. The robust BL-KSF model

Theorem 3.1 *The robust model of robust BL-KSF can be transformed into a Second-order cone optimization problem as follows:*

$$\begin{aligned} &\max_{x} (\hat{\mu}_{BL}^T - r_f e^T)x - \sqrt{\gamma_1 m} A + r_f \\ &s.t. (F_{ED}\sqrt{\gamma_2} + \sqrt{\gamma_1 m}) \| \hat{\Sigma}_{BL}^{\frac{1}{2}} x \|_2 \le (\hat{\mu}_{BL}^T - r_f e^T)x - (m - r_f) \\ &\| \hat{\Sigma}_{BL}^{\frac{1}{2}} x \|_2 \le A \\ &x \in X \end{aligned} \tag{3.1}$$

Proof. We use the methods mentioned by Chen and He [20], here are the proofs. For $r_{BL}\ ED_k\left(\bar{\mu}_{BL}, \bar{\Sigma}_{BL}, g_n(\cdot; q(\mathbf{v}))\right)$, we have

$(r_{BL}^T - r_f e^T)x\ ED_k\left(x^T \bar{\mu}_{BL} - r_f, x^T \bar{\Sigma}_{BL} x, g_n(\cdot; q(\mathbf{v}))\right)$, Then we get

$$P\left((r_{BL}^T - r_f e^T)x < m - r_f\right) \le \alpha \Leftrightarrow \frac{(\bar{\mu}_{BL}^T - r_f e^T)x - (m - r_f)}{\sqrt{x^T \bar{\Sigma}_{BL} x}} \ge F_{ED_k}^{-1}(1-\alpha)$$

In which, $F_{ED_k}(\cdot)$ means cumulative distribution function of $(r_{BL}^T - r_f e^T)x$, and $F_{ED_k}^{-1}(\alpha) < 0$. So the restriction

$$\max_{P\in D_{BL}(\gamma_1,\gamma_2)} P\left((r_{BL}^T - r_f e^T)x < m - r_f\right) \le \alpha$$

can be wrote as

$$\max_{P\in D_{BL}(\gamma_1,\gamma_2)} \{F_{ED_k}^{-1}(\alpha)\sqrt{x^T \bar{\Sigma}_{BL} x} - (\bar{\mu}_{BL}^T - r_f e^T)x\} \le r_f - m \tag{3.2}$$

Then we can rewrite robust BL-KSF model

$$\begin{aligned} &\max_{x} \min_{P\in D_{BL}(\gamma_1,\gamma_2)} \bar{\mu}_{BL}^T x + \left(1 - e^T x\right) r_f \\ &\text{s.t.} \max_{P\in D_{BL}(\gamma_1,\gamma_2)} \{F_{ED_k}^{-1}(\alpha)\sqrt{x^T \bar{\Sigma}_{BL} x} - (\bar{\mu}_{BL}^T - r_f e^T)x\} \le r_f - m \\ &\quad x \in X \end{aligned} \tag{3.3}$$

Let $F_{ED_k}^{-1}(1-\alpha) = F_{ED}$, after the simplification,

$$\max_{x} \min_{P\in D_{BL}(\gamma_1,\gamma_2)} \bar{\mu}_{BL}^T x + \left(1-e^T x\right) r_f$$

$$\text{s.t.} \max_{P\in D_{BL}(\gamma_1,\gamma_2)} \{F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x} - (\bar{\mu}_{BL}^T - r_f e^T)x\} \le r_f - m \tag{3.4}$$

$$x \in X$$

Now we consider the restriction under the ambiguity set.

$$\max_{P\in D_{BL}(\gamma_1,\gamma_2)} \{F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x} - (\bar{\mu}_{BL}^T - r_f e^T)x\} \le r_f - m \tag{3.5}$$

And (3.5) is equal to

$$\max_{P\in D_{BL}(\gamma_1,\gamma_2)} \{F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x} - \min_{P\in D_{BL}(\gamma_1,\gamma_2)} (\bar{\mu}_{BL}^T - r_f e^T)x\} \le r_f - m. \tag{3.6}$$

The ambiguity set $U(\hat{\mu}_{BL},\hat{\Sigma}_{BL})$

$$U(\hat{\mu}_{BL},\hat{\Sigma}_{BL}) = \left\{ (\bar{\mu}_{BL},\bar{\Sigma}_{BL}) \in R^n \times S^n : \begin{array}{c} (\bar{\mu}_{BL}-\hat{\mu}_{BL})^T \hat{\Sigma}_{BL}^{-1} (\bar{\mu}_{BL}-\hat{\mu}_{BL}) \le \gamma_1 m \\ \bar{\Sigma}_{BL} \preceq \gamma_2 \hat{\Sigma}_{BL} \end{array} \right\}$$

$$\max_{\bar{\Sigma}\in U_{\hat{\Sigma}}} F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x} - \min_{\bar{\mu}\in U_{\hat{\mu}}} (\bar{\mu}_{BL}^T - r_f e^T)x \le r_f - m \tag{3.7}$$

$\bar{\Sigma}_{BL}$ and $\bar{\mu}_{BL}$ are independent of each other, so $\max_{P\in D_{BL}(\gamma_1,\gamma_2)} F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x}$ and $\min_{P\in D_{BL}(\gamma_1,\gamma_2)} (\bar{\mu}_{BL}^T - r_f e^T)x$ are affected only by $\bar{\Sigma}_{BL}$ and $\bar{\mu}_{BL}$ correspondingly. We can consider the value of $\bar{\Sigma}_{BL}$ and $\bar{\mu}_{BL}$ separately. And we have For $\min_{\bar{\mu}\in U_{\hat{\mu}}} (\bar{\mu}_{BL}^T - r_f e^T)x$, we consider $\bar{\mu}_{BL-wc}$, the worst case of $\bar{\mu}_{BL}$ in the ambiguity set $U(\hat{\mu}_{BL},\hat{\Sigma}_{BL})$.

$$\begin{aligned} \bar{\mu}_{BL-wc} &= \hat{\mu}_{BL} - \frac{\sqrt{\gamma_1 m}\hat{\Sigma}_{BL}x}{\sqrt{x^T\hat{\Sigma}_{BL}x}} \\ \min_{\bar{\mu}\in U_{\hat{\mu}}} (\bar{\mu}_{BL}^T - r_f e^T)x &= \bar{\mu}_{BL-wc}^T x - r_f e^T x \\ &= (\hat{\mu}_{BL}^T - r_f e^T)x - \sqrt{\gamma_1 m}\sqrt{x^T\hat{\Sigma}_{BL}x} \end{aligned} \tag{3.8}$$

For $\max_{\bar{\Sigma}\in U_{\hat{\Sigma}}} F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x}$, let $\tilde{\Sigma}_{BL} = \bar{\Sigma}_{BL} - \hat{\Sigma}_{BL}$, we have

$$\max_{\bar{\Sigma}\in U_{\hat{\Sigma}}} F_{ED}\sqrt{x^T\tilde{\Sigma}_{BL}x + x^T\hat{\Sigma}_{BL}x} \tag{3.9}$$

$$s.t. \tilde{\Sigma}_{BL} \preceq (\gamma_2 - 1)\hat{\Sigma}_{BL}$$

, where

$\max_{\bar{\Sigma}\in U_{\hat{\Sigma}}} F_{ED}\sqrt{x^T\bar{\Sigma}_{BL}x} = \max_{\bar{\Sigma}\in U_{\hat{\Sigma}}} F_{ED}\sqrt{x^T\tilde{\Sigma}_{BL}x + x^T\hat{\Sigma}_{BL}x} = F_{ED}\sqrt{x^T(\gamma_2\hat{\Sigma}_{BL})x}$. So we can translate the former restriction into

$$F_{ED}\sqrt{x^T(\gamma_2\hat{\Sigma}_{BL})x}-(\hat{\mu}_{BL}^T-r_fe^T)x+\sqrt{\gamma_1 m}\sqrt{x^T\hat{\Sigma}_{BL}x}\le r_f-m \tag{3.10}$$

And we can also get the equivalent objective function

$$\max_x\left(\hat{\mu}_{BL}^T-r_fe^T\right)x-\sqrt{\gamma_1 m}\sqrt{x^T\hat{\Sigma}_{BL}x}+r_f \tag{3.11}$$

The robust BL-KSF model can be rewrote as:

$$\begin{aligned}&\max_x(\hat{\mu}_{BL}^T-r_fe^T)x-\sqrt{\gamma_1 m}\sqrt{x^T\hat{\Sigma}_{BL}x}+r_f\\&s.t.(F_{ED}\sqrt{\gamma_2}+\sqrt{\gamma_1 m})\sqrt{x^T\hat{\Sigma}_{BL}x}\le(\hat{\mu}_{BL}^T-r_fe^T)x-(m-r_f)\\&\quad x\in X\end{aligned} \tag{3.12}$$

IV. Simulation analysis

4.1 Data

we use an example with eight assets, and the data generated by Monte Carlo simulation. We assume that there are 8 assets in the market. The descriptive statistical analysis of risky assets are shown in Table 4.2. The covariance of 8 assets D and the covariance of the market portfolio Σ are shown in Table 1 and Table 2 respectively.

First of all, we denote matrix P and Ω_H as follows:

$$P=\begin{bmatrix}1&&&&&&&-1\\&1&&&&&-1&\\&&1&&&-1&&\\&&&1&&&&\end{bmatrix},\qquad \Omega_H=\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix},$$

$\mathbf{v}=\left[0.003,0.002,0.004,0.005\right]^T,\Pi=\mu-r_fe.$ Then we can use these data to calculate μ_{BL} and Σ_{BL}

$$\mu_{BL}=\Pi+DP^T\left(\Omega+P\Sigma P^T\right)^{-1}(\mathbf{v}-P\Pi) \tag{4.1}$$

$$\Sigma_{BL}=\Sigma-DP^T\left(\Omega+PDP^T\right)^{-1}PD \tag{4.2}$$

4.2 Calibration of parameters γ_1 and γ_2

We need to choose values for parameters γ_1 and γ_2 to control the degree of ambiguity and to finish the robust or adjusted-robust optimization approach. However, we rarely have complete information on the distribution of asset returns in reality. It is crucial to a formal rule to guide an investor in making an appropriate choice of parameters in decisions based on a few historical samples.

Descriptive statistical analysis of risky assets.

	Mean	Median	Maximum	Minimun	Standard Deviation
asset1	0.009603	0.004345	0.245226	-0.24532	0.080807

asset 2	0.012575	0.018006	0.198816	-0.2573	0.076029
asset 3	0.010392	-0.00144	0.273072	-0.25994	0.08861
asset 4	0.006573	-0.0026	0.241028	-0.29088	0.086733
asset 5	0.015177	0.013096	0.253678	-0.25542	0.082706
asset 6	0.004399	0.004072	0.205778	-0.26577	0.070992
asset 7	0.01169	0.00279	0.322148	-0.27956	0.103158
asset 8	0.007674	0.0077	0.276218	-0.27057	0.087391

Table 1: The covariance of 8 assets D.

	asset1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8
asset 1	0.006638	0.005171	0.005267	0.006354	0.006057	0.004686	0.007091	0.005587
asset 2	0.005171	0.005783	0.004125	0.005676	0.005354	0.003996	0.005797	0.00513
asset 3	0.005267	0.004125	0.008451	0.005664	0.005581	0.00528	0.006486	0.005316
asset 4	0.006354	0.005676	0.005664	0.007819	0.006327	0.005115	0.007746	0.006623
asset 5	0.006057	0.005354	0.005581	0.006327	0.007243	0.00467	0.006555	0.005787
asset 6	0.004686	0.003996	0.00528	0.005115	0.00467	0.005106	0.006041	0.004841
asset 7	0.007091	0.005797	0.006486	0.007746	0.006555	0.006041	0.011204	0.007506
asset 8	0.005587	0.00513	0.005316	0.006623	0.005787	0.004841	0.007506	0.007706

Table 2: The covariance of the market portfolio Σ.

	asset 1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8
asset 1	0.000097	0.000062	0.000086	0.000092	0.000092	0.000058	0.000137	0.000092
asset 2	0.000062	0.000056	0.000055	0.000066	0.000066	0.000040	0.000091	0.000069
asset 3	0.000086	0.000055	0.000154	0.000091	0.000094	0.000072	0.000139	0.000098
asset 4	0.000092	0.000066	0.000091	0.000111	0.000094	0.000062	0.000147	0.000108
asset 5	0.000092	0.000066	0.000094	0.000094	0.000114	0.000059	0.000131	0.000099
asset 6	0.000058	0.000040	0.000072	0.000062	0.000059	0.000053	0.000098	0.000067
asset 7	0.000137	0.000091	0.000139	0.000147	0.000131	0.000098	0.000284	0.000163
asset 8	0.000092	0.000069	0.000098	0.000108	0.000099	0.000067	0.000163	0.000143

We can apply bootstrapping procedure under standard assumptions concerning the time series of the returns. This approach that draw random observations with replacement from the available observations is reasonable from a statistical viewpoint, and its computational efficiency is attractive.

1. Construct 120 columns return data of 8 assets as $r_{8\times120}$. Calculate mean $\hat{\mu}$, $\hat{D}, \hat{\mu}_{BL}$ and $\hat{\Sigma}_{BL}$.
2. Draw 80 columns of $r_{8\times120}$ randomly as the i th bootstrapping sample $r_{B_i(8\times80)}$
3. Use $r_{B_i(8\times80)}$ to calculate $\hat{\mu}_{B_i}, \hat{D}_{B_i}, \hat{\mu}_{BL-B_i}$ and $\hat{D}_{BL-B_i}$
4. Let $\gamma_{1-i} = \left(\hat{\mu}_{BL-B_i} - \hat{\mu}_{BL}\right)^{\mathrm{T}} \hat{\Sigma}_{BL}^{-1} \left(\hat{\mu}_{BL-B_i} - \hat{\mu}_{BL}\right)$ and $\gamma_{2-i} = \dfrac{\| \hat{\Sigma}_{BL-B_i} \|}{\| \hat{\Sigma}_{BL} \|}$. For i th bootstrapping

procedure, we can get γ_{1-i} and γ_{2-i}.

5. Repeat the above step 1 to 4 for a total of 300 times, find 300 pairs of ambiguity set parameters γ_{1-i}

and γ_{2-i}, sort them in ascending order, and select the 95 % quantile value of the the γ_{1-i} and γ_{2-i} sequence as the best estimates of the two parameters , $\hat{\gamma}_1$ and $\hat{\gamma}_2$.

Through the above experiments, the following results are obtained:

$$\hat{\gamma}_1 = 0.05638$$

$$\hat{\gamma}_2 = 0.03324$$

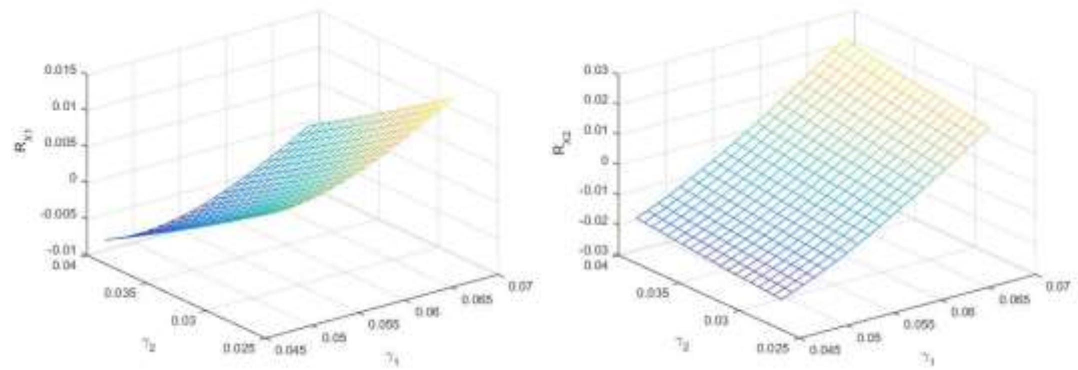
So far, the estimation of all parameters in the second-order cone optimization model has been determined. The SeDuMi.1.3 toolkit in Matlab will be used to solve the second-order optimization problem.

4.3 Sensitivity analysis to the parameters of the ambiguity set

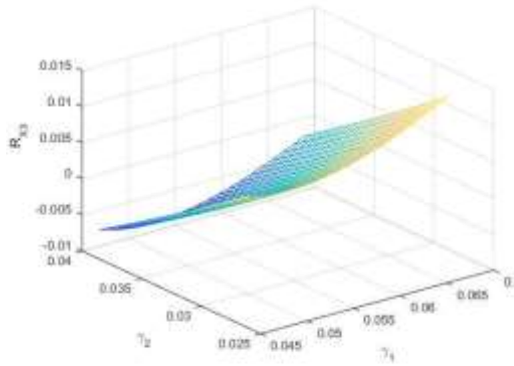
In order to study the specific effect of ambiguity set parameters on each asset share in the optimal solution, the following parameters are defined here:

$$\mathrm{R}_{\mathrm{x}_j} = \frac{\mathrm{x}_j^*\left(\gamma_1,\gamma_2\right) - \mathrm{x}_j^*\left(\hat{\gamma}_1,\hat{\gamma}_2\right)}{\mathrm{x}_j^*\left(\hat{\gamma}_1,\hat{\gamma}_2\right)}, j = 1,2,\cdots 8 \tag{4.3}$$

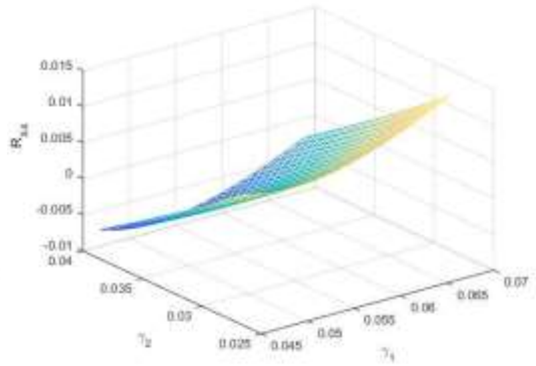
Among them, $\gamma_1 \in (0.04510, 0.06766), \gamma_2 \in (0.02659, 0.03989)$. $\mathrm{x}_j^*\left(\gamma_1,\gamma_2\right)$ indicates the share of assets in the optimal solution of the second-order cone optimization problem with the investor's point of view,when the ambiguity set parameter is γ_1 and γ_2. From this parameter, the sensitivity of each asset to the ambiguity set parameters and the optimal solution can be expressed as $\mathrm{R}_{\mathrm{x}_j}$. The estimated values of ambiguity set parameters have been given in the previous section, $\hat{\gamma}_1 = 0.05638, \hat{\gamma}_2 = 0.03324$. The results obtained are shown in the Fig. 1.



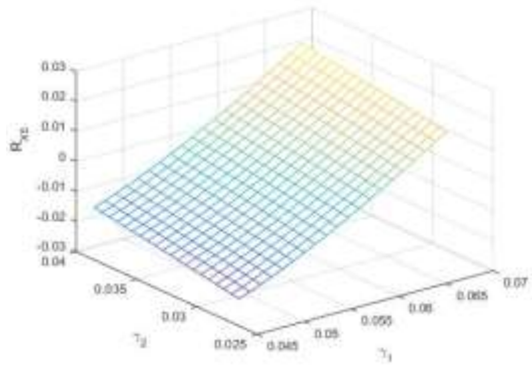
(a) (b)



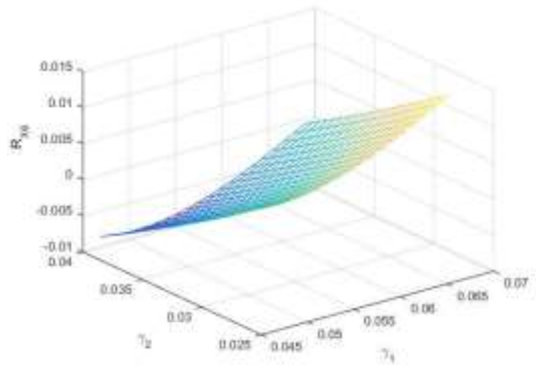
(c)



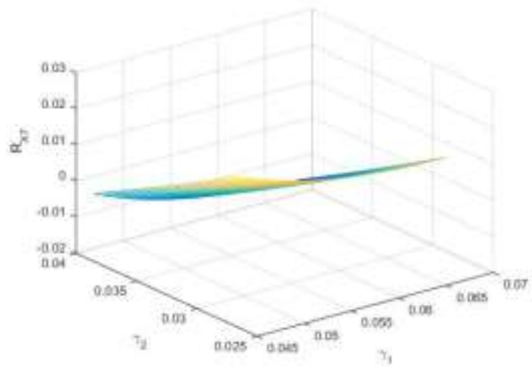
(d)



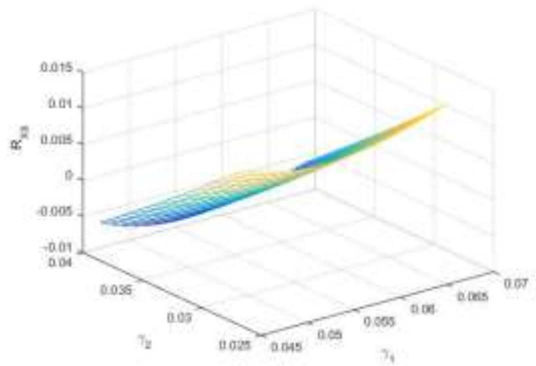
(e)



(f)



(g)



(h)

Figure 1: Sensitivity analysis of ambiguity set parameters in the optimal solution of the model. Relative difference R_{x_j} (i=1,...,8 in percentage) for varying parameters γ_1 and γ_2.

4.4 Effective frontier and robustness analysis of optimal solution

According to the various parameters calculated in Section 4.1, including the return on risk assets $\hat{\mu}_{BL}$ and covariance with the investor's view $\hat{\Sigma}_{BL}$, the ambiguity set parameters estimated by using the Bootstrap method $\hat{\gamma}_1 = 0.05638, \hat{\gamma}_2 = 0.03324$, as well as the return on risk assets $r_f = 0.003$ and the safe return on investors $m = 0.005$. In order to obtain the effective frontier of the BL-KSF ,the risk aversion coefficient α is changing here. The smaller the α, the greater the $F_{ED_k}^{-1}(1-\alpha) = F_{ED}$, which is the higher the risk aversion coefficient of investors. Let α vary from 0.01 to 0.20 to find the variety of the optimal portfolio return.

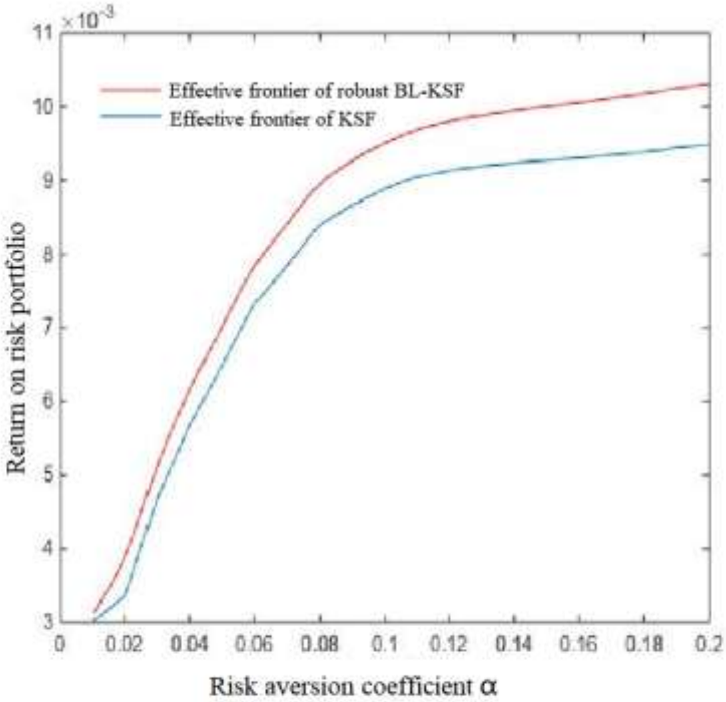


Figure 2: The efficient frontiers of KSF and robust BL-KSF.($\hat{\gamma}_1 = 0.05638, \hat{\gamma}_2 = 0.03324, r_f = 0.003, m = 0.005$)

The red curve in the Fig. 2 represents the effective frontier of the robust BL-KSF model when the return of risk assets with investor views is uncertain, and the blue curve represents the effective boundary of the KSF model when the risk asset returns follow elliptic distribution.

Obviously, the effective frontier of the robust BL-KSF model is smoother, and the whole is located above the effective frontier of the KSF model. It shows that under the same degree of risk aversion, the optimal solution of the robust model has a higher portfolio return rate. In other words, corresponding to the same level of portfolio return, the optimal solution of the robust model can satisfy a more stringent level of risk aversion. Therefore, we can assume that the robust BL-KSF model under under distribution ambiguity has better robustness than the KSF model.

V. Conclusion

In this paper,we use the black-litterman model to quantify investor views, and show that BL-KSF risk measure over distributional ambiguity sets can be computed efficiently via conic optimization techniques. We also introduce bootstrapping method for calibrating the levels of ambiguity based to provides an important modeling guidance and may be of interest to practitioners. By using simulated market data, we get the results of numerical experiments and it demonstrate that our robust methods can construct more diversified portfolios which are superior to their non robust counterparts in terms of portfolio stability. We know that a number of ambiguity sets on probability distributions under uncertainty have been proposed, however, to the best of our knowledge, there seems to be no consensus on whether or not the robust BL-KSF optimization solution under an ambiguity set is intrinsically better than the one under an alternative one. It remains an interesting topic that deserves to be investigated in future.

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