International Journal of Business Marketing and Management (IJBMM)
Volume 5 Issue 3 March 2020, P.P. 67-80
ISSN: 2456-4559
www.ijbmm.com

Asian option on Riemannian manifolds

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Abstract : *Zhang [3] studied the European option pricing problem when the underlying asset follows the geometric Riemannian Brownian motion. Motivated by [3], we, in this paper, investigate the asian option on Riemannian manifolds. By exploring the relationship between Riemannian Brownian motion on Riemannian manifolds and Euclidean Brownian motion, we derive the pricing equation of the geometric average asian option on Riemannian manifolds, and provide a semi-explicit expression for the solution by using the fundamental solution.*

MR*: 35Q30; 91B24*
Keywords: *Riemannian manifolds, Riemannian metric, Riemannian Brownian motion, stochastic development, Asian option pricing.*

I. INTRODUCTION

In 1973, under some assumptions, Black and Scholes[7] obtained a relatively complete option pricing formula (Black-Scholes formula) by constructing the stochastic differential equations of the underlying asset price and applying the method of Risk Hedging. However, too many assumptions result in some errors between the option value obtained from the Black-Scholes formula and the actual data in the financial market. In order to solve this problem, researchers made further research in combination with the real market, most of them relaxed the assumptions of the Black-Scholes model. In the assumptions of Black- Scholes model, the change of stock price is continuous, which means the diffusion process of stock price obeys lognormal distribution. But in the real world, there will always be some important information that leads to dramatic change in stock price process, and then the stock price will change intermittently, such as jumping. Based on this consideration, in 1975, Merton[18] established a different diffusion process of stock price, namely jump diffusion model. In this model, Merton added the position process to the option pricing model. In 1987, hull and white[4] approximated the average value of each node of the binary treeby adding nodes and using the method of linear interpolation, and finally calculated the option price by the method of backward discount.

In 1990, Kemna and Vorst[25] obtained an analytic formula of geometric average option pricing by changing the volatility of asset price and the execution price of option contract. In 1991, Elias Stein and Jeremy Stein[14] assumed that volatility was driven by the arithmetic process of Ornstein-Uhlenbeck, and derived the option pricing by double integration. In 1992, Edmond lévy[23] used geometric Brownian motion to describe the arithmetical average distribution of price and transformed the pricing problem of Asian option into that of European option so that a more accurate approximation was obtained. In 1998, after further exploration, Varikooty, Jha and Chalasani[16] solved the pricing problem of Asian option in the discrete case by using the trident tree method. In 1999, Chan[11] replaced Brownian motion with a general Lévy process, and obtained an integral differential equation about the option price.

In 2001, Jin E. Zhang[19] obtained a semi-explicit expression of the pricing formula of the arithmetic mean Asian option with fixed strike price. In addition, he calculated it by numerical method and got good numerical results. The expressions of solutions in some feasible regions do exist, which, however, was not fully utilized. In 2002, Ju[24] put forward a complex option pricing method. He used the average characteristic function of Taylor expansion to get the approximate solution of the pricing problem, which put aside the relevant assumptions of Black-Scholes model.

Since the Black-Scholes model was put forward, scholars from the whole world have put forward various options pricing models and methods, such as PDE method, analytic approximation method, binary tree method, finite difference method, Monte-Carlo simulation method, etc. It is quite surprising that in 2016, Zhang[3] studied the pricing of European option whose underlying asset diffusion process follows the general geometric Riemannian Brownian motion, obtained the corresponding semi-explicit solution. By choosing proper Riemannian metrics, he verified that the distribution of return rates of the stock has the character of leptokurtosis and fat-tail, and explained option pricing bias and implied volatility smile.

In fact, besides time, there are many factors that affect the process of asset pricein the real world besides time, such as exchange rate, inflation, policy implementation, etc. these unknown factors could be regarded as a function $\gamma(t)$ of time. We consider the asset price function in the sense of $S = S(t, \gamma(t))$. For convenience, we assume $S = S(\gamma(t))$. However $\gamma(t)$ is not necessarily linear with time *t*. The line space is "curved", we need to introduce the concept of manifolds. In this paper, we assume that the asse follows Riemannian Brownian motion and study the pricing of Asian option on Riemannian manifolds.

II. PREPARATORY

According to the need of studying Asian option pricing on Riemannian manifolds, this section introduces some necessary knowledge of stochastic differential geometry. For details, please refer to [2, Chapter2, Chapter3, 3, Chapter3].

Let *M* be a d-dimensional smooth differential manifold, $T_b M$ be the tangent space at *b*, *TM* be the tangent bundle, $F(M)$ be *M's* standard frame bundle, and $\pi : F(M) \to M$ denote a projection map. If the vector field $u_t e$ is parallel along πu_t for each $e \in R^d$, then the curve u_t in $F(M)$ is horizontal, $u'(0)$ is called the horizontal lift of the tangent vector $(\pi u)'(0)$.

Let $e_i \in R^d, i = 1,2,\cdots,d$ be the coordinate unit vectors. We define the vector fields $H_i, i = 1,2,\cdots,d$, by

$$H_i(u) := \text{ the horizontal lift of } ue_i \in T_{\pi u} M \to u .$$

where $u \in F(M)$.

Let u_t be a horizontal lift of a differentiable curve b_t on *M*. Since $\dot{b}_t \in T_{b_t} M$, we have $u_t^{-1} \dot{b}_t \in R^d$. The anti-development of the curve b_t (or of the horizontal curve u_t) is a curve w_t in R^d defined by

$$w_t = \int_0^t u_s^{-1} \dot{b}_s ds \, . \tag{2.1}$$

Hence the anti-development w_t and the horizontal lift u_t of a curve b_t on *M* are connected by an ordinary differential equation on $F(M)$:

$$\dot{u}_t = H_i(u_t)\dot{w}_t^i \, . \tag{2.2}$$

Let ∇ be an affine connection defined on the tangent bundle *TM*. We consider the following SDE on the frame bundle $F(M)$ in the sense of Stratonovish integral:

$$dU_t = H_i(U_t) \circ dW_t^i \, , \tag{2.3}$$

where *W* is an R^d -valued semimartingale. Stratonovich integral has the advantage of leading to ordinary chain rule formulas under a transformation, i.e. there are no second order terms in the Stratonovich analogue of the *Itö* transformation formula. This property makes the Stratonovich integral natural to use for example in connection with stochastic differential equations on manifolds. We now give some definitions. All processes are defined on a filtered probability space $(\Omega, \mathcal{F}_*, P)$ and are $\mathcal{F}_*$ -adapted.

Definition 2.1 *([2, Definition 2.3.1])* (i)An $F(M)$- valued semimartingale *U* is said to be horizontal if there exists an R^d - valued semimartingale *W* such that (2.3) holds. The unique *W* is called the anti-development of *U* (or of its projection $B = \pi U$).

(ii) Let *W* be an R^d - valued semimartingale and U_0 be an $F(M)$- valued, $\mathcal{F}_0$- measurable random variable. The solution of the SDE (2.3) is called a (stochastic) development of *W* in $F(M)$. Its projection $B = \pi U$ is called a (stochastic) development of *W* in *M*.

(iii) Let *B* be an *M*-valued semimartingale. An $F(M)$- valued horizontal semimartingale *U* such that its projection $\pi U = B$ is called a (stochastic) horizontal lift of *B*.

Assume that M is a closed submanifold of R^d and regard $B = \{B^\alpha\}$ as an R^d - valued semimartingale. For each $b \in M$, let $P(b): R^d \to T_b M$ be the orthogonal projection from R^d to the subspace $T_b M \subseteq R^d$. Then intuitively we have the horizontal lift *U* of *B* is the solution of the following equation on F(M)

$$dU_t = P_\alpha^*(U_t) \circ dB_t \, , \tag{2.4}$$

where $P_\alpha^*(u)$ is the horizontal lift of $P_\alpha(\pi u)$. And the anti-development *W* of horizontal semimartingale *U* satisfies

$$W_t = \int_0^t U_s^{-1} P_\alpha(B_s) \circ dB_s^\alpha \, , \tag{2.5}$$

where $B_t = \pi U_t$.

Unlike the Euclidean Brownian motion, Brownian motion on a Riemannian manifold *M* is a diffusion process generated by $\Delta_M / 2$, where Δ_M is the Laplace-Beltrami operator on *M*. We assume that *M* is a Riemannian manifold equipped with the Levi-Civita connection ∇ , Given a probability measure μ on *M*, there is a unique $\Delta_M/2$-diffusion measure P_μ on the filtered measurable space $(W(M), B_*)$(the path space over *M*). Any $\Delta_M/2$-diffusion measure on $W(M)$is called a Wiener measure on $W(M)$. In general, an *M*-valued stochastic process *B* is a measurable map (random variable) $B:\Omega \to W(M)$defined on some measurable space $(\Omega, \mathcal{F})$. Rougly speaking, Brownian motion on *M* is any *M*-valued stochastic process *B* whose law is a Wiener measure on the path space $W(M)$.

Proposition2.2 *([2, Proposition 3.2.1])* Let $B:\Omega \to W(M)$be a measurable map defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mu = P \circ B_0^{-1}$ be its initial distribution. Then the following statements are equivalent.

(i) *B* is a $\Delta_M/2$-diffusion process (a solution to the martingale problem for $\Delta_M/2$with respect to its own filtration $\mathcal{F}_*^B$), i.e.

$$M^f(B)_t \overset{def}{=} f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta_M f(B_s)\,ds, 0 \le t < e(B)$$

is an $\mathcal{F}_*^B$-local martingale for all $f \in C^\infty(M)$.

(ii) The law $P^B = P \circ B^{-1}$is a Wiener measure on $W(M)$ i.e. $P^B = P_\mu$.

(iii) *B* is a $\mathcal{F}_*^B$-semimartingale on *M* whose anti-development is a standard Euclidean Brownian motion.

An *M*-valued process *B* satisfying any of the above conditions is called a (Riemannian) Brownian motion on *M*.

Let $M = R$ be equipped with a general connection given by $\nabla_e e = \Gamma e$, where *e* is the usual unit vector on *R*: $e(f) = f'$, and $\Gamma \in C^\infty(R)$. We assume $u = (b, y) \in F(R), b_t \in C^\infty(R)$, and $u_t = (b_t, y_t)$ is a horizontal lift of b_t such that $\dot{b}(0) = 1$ and $u_0 = u$. Hence $\nabla_{\dot{b}_t} y_t = 0$, i.e.

$$\dot{y}_t + \Gamma(b_t)\dot{b}_t y_t = 0.$$

Since the orthogonal projection $P(b): R \to T_b R \cong R$is an identity, the horizontal lift of $P(\pi u)$at *u* is

$$P^*(u) = (\dot{b}(0), \dot{y}(0)) = (1, -y\Gamma(b)).$$

Let *B* be a semimartingale. According to (2.4), the horizontal lift $U_t = (B_t, Y_t)$ of *B* is determined by

$$\begin{cases} d(B_t, Y_t) = (1, -Y_t\Gamma(B_t)) \circ dB_t \\ (B_0, Y_0) = (B_0, y) \end{cases}. \quad (2.6)$$

Note that

$$d\int_0^{B_t} \Gamma(s)ds = \Gamma(B_t) \circ dB_t.$$

The following identity holds,

$$\int_0^t \Gamma(B_s) \circ dB_s = \int_0^{B_t} \Gamma(s)ds - \int_0^{B_0} \Gamma(s)ds. \quad (2.7)$$

Define

$$G(x) = \int_0^x \Gamma(y)dy, \phi(x) = \int_0^x e^{G(y)}dy.$$

From (2.6) and (2.7), we have

$$Y_t = y\exp\left(-\int_0^t \Gamma(B_s)\circ dB_s\right)$$
$$= y\exp\left(G(B_0)\right)\exp\left(-G(B_t)\right).$$

Hence, the horizontal lift which passes through (B_0, y) of B is given by

$$U_t = \left(B_t, y\exp\left(G(B_0)\right)\exp\left(-G(B_t)\right)\right).$$

If the anti-development W_t of B satisfys $W_0 = 0$, combined with (2.5), we have

$$W_t = \int_0^t y^{-1}\exp\left(-G(B_0)\right)\exp\left(G(B_s)\right)\circ dB_s$$
$$= y^{-1}\exp\left(-G(B_0)\right)\left(\varphi(B_t)-\varphi(B_0)\right). \tag{2.8}$$

Theorem 2.3 Let $M = R$ be equipped with a Riemannian metric *g*. If *B* is a Riemannian Brownian motion and the horizontal lift *U* of *B* satisfies $U_0 = (B_0, y)$, then

$$dB_t = y\sqrt{\frac{g(B_0)}{g(B_t)}}dW_t - \frac{1}{4}y^2 g(B_0)\frac{g'(B_t)}{g(B_t)^2}dt. \tag{2.9}$$

Proof: refer to [3] to give a simple proof as follows:

There is a unique Levi-Civita connection compatible with metric *g* on Riemannian manifold (R, g). In local coordinates, the Christoffel sign of Levi-Civita connection induced by *g* is given by

$$\Gamma(x) = \frac{1}{2}g(x)^{-1}\frac{\partial g}{\partial x}(x) \quad = \frac{1}{2}\frac{\partial}{\partial x}\left(\ln\left(g(x)\right)\right),$$

where $g(x)^{-1}$ is the inverse matrix of $g(x)$.

We have

$$G(x) = \ln\sqrt{\frac{g(x)}{g(0)}}, \varphi(x) = \frac{1}{\sqrt{g(0)}}\int_0^x \sqrt{g(s)}ds.$$

Then through (2.8), we have

$$dW_t = y^{-1}\exp\left(-G(B_0)\right)\circ d\varphi(B_t)$$
$$= y^{-1}\exp\left(-\ln\sqrt{\frac{g(B_0)}{g(0)}}\right)\sqrt{\frac{g(B_t)}{g(0)}}\circ dB_t$$
$$= y^{-1}\sqrt{\frac{g(B_t)}{g(B_0)}}\circ dB_t.$$

If *B* is a Riemannian Brownian motion, then

$$dB_t = y\sqrt{\frac{g(B_0)}{g(B_t)}} \circ dW_t$$

$$= y\sqrt{\frac{g(B_0)}{g(B_t)}}dW_t + \frac{1}{2}y\sqrt{g(B_0)}d\left(\frac{1}{\sqrt{g(B_t)}}\right)dW_t$$

$$= y\sqrt{\frac{g(B_0)}{g(B_t)}}dW_t - \frac{1}{4}y^2 g(B_0)\frac{g'(B_t)}{g(B_t)^2}dt .$$

In particular, it is pointed out in reference [3] that if the initial value of the horizontal lift U is fixed and satisfies

$$U_0 = \left(B_0, \frac{1}{\sqrt{g(B_0)}}\right),$$

then

$$dB_t = \frac{1}{\sqrt{g(B_t)}}dW_t - \frac{1}{4}\frac{g'(B_t)}{g(B_t)^2}dt . \qquad (2.10)$$

If M is equipped with a Riemannian metric, we can define the anti-development, horizontal lift, etc. Based on this point of view, (2.10) is considered on $O(R)$. Since $g(x)^{-\frac{1}{2}}$: (R,the usuall Euclidean metric) $\rightarrow (T_x R, g(x))$ is unitary.

III. PRICING MODEL

The purpose of this section is to establish the pricing equation of Asian option on Riemannian Manifolds (taking geometric average call option with fixed strike price as an example).

Let B be a Riemannian Brownian motion on (R, g), where $B_0 = 0$. $\{\mathcal{F}_t\}_{t\geq 0}$ is a natural filtration generated by B, and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a probability space. Let us consider a simple market. In this market, there are two assets,the stock S and the risk-free bond D, whose prices satisfy the following diffusion processes

$$dS_t = \mu S_t dt + \sigma S_t \circ dB_t , \qquad (3.1)$$

$$dD_t = rD_t dt ,$$

where μ, σ and r are some constants.

Combine with $It\ddot{o}$ -Stratonovich integral conversion formula, we have

$$dB_t = -\frac{\mu}{\sigma}dt - \frac{\sigma}{2}d\langle B\rangle_t + \frac{1}{\sigma S_t}dS_t . \qquad (3.2)$$

From (2.10), we have

$$d\langle B\rangle_t = \frac{1}{g(B_t)}dt . \qquad (3.3)$$

Substitute (3.2), (2.10) into (3.1), we have

$$\frac{dS_t}{S_t} = \left(\mu + \frac{\sigma^2}{2g(B_t)} - \frac{\sigma}{4}\frac{g'(B_t)}{g(B_t)^2}\right)dt + \frac{\sigma}{\sqrt{g(B_t)}}dW_t, \tag{3.4}$$

where W is the anti-development of B, which is a standard Euclidean Brownian motion.

Theorem 3.1 Define a process by

$$\tilde{W}_t := W_t - \int_0^t \Theta_u du,$$

where

$$\Theta_u := \sqrt{g(B_t)}\left(\frac{r-\mu}{\sigma} - \frac{\sigma}{2g(B_t)} + \frac{1}{4}\frac{g'(B_t)}{g(B_t)^2}\right). \tag{3.5}$$

Let

$$Z_t = \exp\left\{\int_0^t \Theta_u dW_u - \frac{1}{2}\int_0^t \Theta_u^2 du\right\}.$$

Assume $E\int_0^T \Theta^2(u) Z_u^2 du < \infty$, then the probability measure $\tilde{P}$ given by

$$\tilde{P}(A) = \int_A Z_\omega dP(\omega), \forall A \in \mathcal{F}$$

is a probability measure on $(\Omega, \mathcal{F}_T)$, and the process $\tilde{W}_t$ is a standard Euclidean Brownian motion under $\tilde{P}$.

Proof: referring to [3], applying Girsanov theorem ([13, theorem 5.3.1]) and Novikov theorem (12, proposition 5.12), we can prove theorem 3.1.

Suppose we have used the risk neutral measure to complete the conversion of the stock price process, put

$$\tilde{W}_t := W_t - \int_0^t \Theta_u du$$

into (3.4), we have

$$\frac{dS_t}{S_t} = \left(\mu + \frac{\sigma^2}{2g(B_t)} - \frac{\sigma}{4}\frac{g'(B_t)}{g(B_t)^2} + \frac{\sigma}{\sqrt{g(B_t)}}\Theta_t\right)dt + \frac{\sigma}{\sqrt{g(B_t)}}d\tilde{W}_t,$$

Combining with (3.5), we have

$$\frac{dS_t}{S_t} = rdt + \frac{\sigma}{\sqrt{g(B_t)}}d\tilde{W}_t,$$

$$dB_t = \left(\frac{r-\mu}{\sigma} - \frac{\sigma}{2g(B_t)}\right)dt + \frac{1}{\sqrt{g(B_t)}}d\tilde{W}_t.$$

Let's choose the risk-free bond D with one unit payoff as a numeraire. For any process Z, let us denote the numeraire-rebased process $D^{-1}Z$ by Z^D. Then the price process can be recorded as S_t^D, i.e.

$$S_t^D = \frac{S_t}{D_t},$$

then

$$\frac{dS_t^D}{S_t^D} = \left(\mu - r + \frac{\sigma^2}{2g(B_t)} - \frac{\sigma}{4} \frac{g'(B_t)}{g(B_t)^2} \right) dt + \frac{\sigma}{\sqrt{g(B_t)}} dW_t .$$

Hence under $\tilde{P}$, we have

$$\frac{dS_t^D}{S_t^D} = \left(\mu - r + \frac{\sigma^2}{2g(B_t)} - \frac{\sigma}{4} \frac{g'(B_t)}{g(B_t)^2} + \frac{\sigma}{\sqrt{g(B_t)}} \Theta_t \right) dt + \frac{\sigma}{\sqrt{g(B_t)}} d\tilde{W}_t , \qquad (3.6)$$

$$\frac{dS_t^D}{S_t^D} = \frac{\sigma}{\sqrt{g(B_t)}} d\tilde{W}_t .$$

Since $\tilde{W}_t$ is a standard Euclidean Brownian motion under $\tilde{P}$, S_t^D is a martingale under $\tilde{P}$.

From (3.6), we have

$$S_t^D = S_0^D \exp\left\{ \int_0^t \frac{\sigma}{\sqrt{g(B_u)}} d\tilde{W}_u - \frac{1}{2} \int_0^t \frac{\sigma^2}{g(B_u)} du \right\},$$

Noting that

$$S_t^D = S_t , S_0^D = \exp(rt) S_0 ,$$

the stock price under $\tilde{P}$ is

$$S_t = \exp(rt) S_0 \exp\left\{ \int_0^t \frac{\sigma}{\sqrt{g(B_u)}} d\tilde{W}_u - \frac{1}{2} \int_0^t \frac{\sigma^2}{g(B_u)} du \right\}.$$

From (3.1), we have $\mathcal{F}_t^S \subset \mathcal{F}_t$,and

$$dB_t = -\frac{\mu}{\sigma} dt + \frac{1}{\sigma S_t} \circ dS_t ,$$

where $\sigma > 0$, i.e. $\mathcal{F}_t \subset \mathcal{F}_t^S$. Hence $\mathcal{F}_t^S = \mathcal{F}_t$. However we can't assert that V_t is a function of time *t* and stock price S_t by Markov property. Since V_T is not a function of *T* and S_T, it also relate to the path of *S*.

In order to solve the expression of Asian option price on Riemannian manifold, we extend S_t and introduce the second process

$$J_t = \exp\left\{ \frac{1}{t} \int_0^t \ln S_\tau d\tau \right\},$$

the SDE of J_t is

$$dJ = J \left(\frac{\ln S - \ln J}{t} \right) dt . \qquad (3.7)$$

Since (S_t, J_t) obey (3.1) and (3.7), they form a two-dimensional Markov process [13,Corollary 6.3.2], the payoff V_T is a function of *T* and the terminal value (S_T, J_T).

In fact, by

$$V_T = (J_T - K)^+,$$

V_T only depends on T and J_T. Therefore, there must be a function $V(t,x,y)$ so that the price of Asian option can be expressed as

$$V(t, S_t, J_t) = \tilde{E}\left[\exp\left\{-\int_t^T r_u du\right\}(J_T - K)^+ \middle| \mathcal{F}_t\right]$$

$$= \tilde{E}\left[\exp\left\{-\int_t^T r_u du\right\} V_T \middle| \mathcal{F}_t\right]. \tag{3.8}$$

Theorem 3.2 The price function $V(t, S_t, J_t)$ of Asian option on Riemannian manifolds in (3.8) satisfies the folloing PDE:

$$\frac{\partial V}{\partial t} + J\frac{\partial V}{\partial J}\frac{\ln S - \ln J}{t} + \frac{1}{2}\frac{\sigma^2 S^2}{g\left(\frac{1}{\sigma}(\ln S - \ln S_0 - \mu t)\right)}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

and boundary condition

$$V(S, J, T) = (J - K)^+.$$

Proof: Suppose $\sigma_t \neq 0$ for each $t \in [0,T]$. Let *V=V(S,J,t)* be a replicable Asian option with the maturity *T* and form a portfolio

$$\Pi = V(S, J, t) - \Delta_t S.$$

According to the method of complete hedging, this portfolio is risk-free. Hence its yield is risk-free yield, i.e.

$$d\Pi = r\Pi dt = r(V - \Delta_t S)dt.$$

From *Itö* formula, we have

$$d\Pi = dV - \Delta_t dS$$

$$= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial J}dJ + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 - \Delta_t dS$$

$$= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\sigma^2 S^2}{g(B_t)}\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial J}\frac{dJ}{dt}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta_t\right)dS. \tag{3.9}$$

In order to make Π risk-free within $(t, t+dt)$, take

$$\Delta_t = \frac{\partial V}{\partial S}.$$

Substituting (3.9) and deleting *dt*, we have

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial J}\frac{dJ}{dt}+\frac{1}{2}\frac{\sigma^2 S^2}{g\left(\frac{1}{\sigma}(\ln S-\ln S_0-\mu t)\right)}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \tag{3.10}$$

where $J_t=\exp\left\{\frac{1}{t}\int_0^t \ln S_\tau d\tau\right\}$.

Hence the pricing model of Asian option on Riemannian manifolds is

$$\begin{cases}\frac{\partial V}{\partial t}+J\frac{\partial V}{\partial J}\frac{\ln S-\ln J}{t}+\frac{1}{2}\frac{\sigma^2 S^2}{g\left(\frac{1}{\sigma}(\ln S-\ln S_0-\mu t)\right)}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \\ V(S,J,T)=(J-K)^+\end{cases}. \tag{3.11}$$

IV. MODEL SOLUTION

In this section, we will deal with the Asian option pricing model (3.11) obtained in section 3 by a series of mathematical methods.

For

$$\frac{\partial V}{\partial t}+J\frac{\partial V}{\partial J}\frac{\ln S-\ln J}{t}+\frac{1}{2}\frac{\sigma^2 S^2}{g\left(\frac{1}{\sigma}(\ln S-\ln S_0-\mu t)\right)}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0,$$

let

$$\xi=\frac{t\ln J+(T-t)\ln S}{\sigma T}, V(S,J,t)=U(\xi,t), \tag{4.1}$$

we have

$$\frac{\partial V}{\partial t}=\frac{\partial U}{\partial t}+\frac{\partial U}{\partial \xi}\left[\frac{\ln J}{\sigma T}-\frac{\ln S}{\sigma T}\right],$$

$$\frac{\partial V}{\partial J}=\frac{\partial U}{\partial \xi}\frac{t}{\sigma TJ}, \quad \frac{\partial V}{\partial S}=\frac{T-t}{\sigma TS}\frac{\partial U}{\partial \xi},$$

$$\frac{\partial^2 V}{\partial S^2}=\left(\frac{T-t}{\sigma TS}\right)^2\frac{\partial^2 U}{\partial \xi^2}-\frac{T-t}{\sigma TS^2}\frac{\partial U}{\partial \xi}.$$

Substituting the above results into (3.9), we have

$$\frac{\partial U}{\partial t}+\frac{1}{2g(\lambda)}\left(\frac{T-t}{T}\right)^2\frac{\partial^2 U}{\partial \xi^2}+\left(\frac{r}{\sigma}-\frac{\sigma}{2g(\lambda)}\right)\left(\frac{T-t}{T}\right)\frac{\partial U}{\partial \xi}-rU=0,$$

where $\lambda=\int_0^t \frac{T}{T-s}\xi'(s)ds-\frac{\mu}{\sigma}t$.

And the boundary condition

$$U\big|_{t=T}=V\big|_{t=T}=(J-K)^+\big|_{t=T}=\left(e^{\sigma\xi}-K\right)^+.$$

Under (4.1), the Asian option pricing problem (3.11) is transformed into the following problem

$$\begin{cases} \dfrac{\partial U}{\partial t}+\dfrac{1}{2g(\lambda)}\left(\dfrac{T-t}{T}\right)^2\dfrac{\partial^2 U}{\partial \xi^2}+\left(\dfrac{r}{\sigma}-\dfrac{\sigma}{2g(\lambda)}\right)\left(\dfrac{T-t}{T}\right)\dfrac{\partial U}{\partial \xi}-rU=0 \\ U\big|_{t=T}=\left(e^{\sigma\xi}-K\right)^+ \end{cases}, \tag{4.2}$$

where $\lambda=\int_0^t \frac{T}{T-s}\xi'(s)ds-\frac{\mu}{\sigma}t$.

Next, let

$$W=Ue^{\alpha(t)},\eta=\xi+\beta(t),\ \ \tau=\gamma(t). \tag{4.3}$$

It's not hard to have

$$\frac{\partial U}{\partial t}=e^{-\alpha(t)}\left[\gamma'(t)\frac{\partial W}{\partial \tau}+\beta'(t)\frac{\partial W}{\partial \eta}-\alpha'(t)W\right],$$

$$\frac{\partial U}{\partial \xi}=e^{-\alpha(t)}\frac{\partial W}{\partial \eta},\ \frac{\partial^2 U}{\partial \xi^2}=e^{-\alpha(t)}\frac{\partial^2 W}{\partial \eta^2}.$$

Substituting the above results into (4.2), we have

$$\gamma'(t)\frac{\partial W}{\partial \tau}+\frac{1}{2g(\lambda)}\left(\frac{T-t}{T}\right)^2\frac{\partial^2 W}{\partial \eta^2}+\left[\left(\frac{r}{\sigma}-\frac{\sigma}{2g(\lambda)}\right)\left(\frac{T-t}{T}\right)+\beta'(t)\right]\frac{\partial W}{\partial \eta}-\left(r+\alpha'(t)\right)W=0. \tag{4.4}$$

Let

$$r+\alpha'(t)=0,\beta'(t)=-\frac{r}{\sigma}\left(\frac{T-t}{T}\right),\gamma'(t)=-\left(\frac{T-t}{T}\right)^2,$$

and the terminal conditions

$$\alpha(T)=\beta(T)=\gamma(T)=0.$$

We have

$$\alpha(t)=r(T-t),\beta(t)=\frac{r}{2\sigma T}(T-t)^2,\gamma(t)=\frac{1}{3T^2}(T-t)^3.$$

Substituting the above values into (4.4) , the problem (4.2) is transformed into

$$\begin{cases} \dfrac{\partial W}{\partial \tau}-\dfrac{1}{2g(\bar\lambda)}\dfrac{\partial^2 W}{\partial \eta^2}+\dfrac{\sigma}{2g(\bar\lambda)}\left(\dfrac{T}{\sqrt[3]{3T^2\tau}}\right)\dfrac{\partial W}{\partial \eta}=0 \\ W(\eta,0)=\left(e^{\sigma\eta}-K\right)^+ \end{cases}. \tag{4.5}$$

where $\bar\lambda(\eta,\tau)=\lambda(\xi,t)$.

For general Riemannian metric g, it is not easy to obtain a explicit solution of (4.5). We here provide a semi-explicit expression for the solution by using the fundamental solution technique.

For

$$\frac{\partial W}{\partial \tau}=\frac{1}{2g\left(\bar{\lambda}\right)}\frac{\partial^2 W}{\partial \eta^2}-\frac{\sigma}{2g\left(\bar{\lambda}\right)}\left(\frac{T}{\sqrt[3]{3T^2\tau}}\right)\frac{\partial W}{\partial \eta}, \tag{4.6}$$

let $\omega(\eta,\tau)$ satisfy the equation

$$\frac{\partial \omega}{\partial \tau}-\frac{1}{2g\left(\hat{\lambda}\right)}\frac{\partial^2 \omega}{\partial \eta^2}=\delta(\eta,\tau)=\delta(\eta)\delta(\tau),$$

where $\hat{\lambda}(y,\kappa)=\bar{\lambda}(\eta,\tau)$, $\delta(\cdot)$ is a Dirac-δ function.

By using the generalized Fourier transform to find ω, we assume that $\omega(\eta,\tau)$ is a tempered distribution with respect to η [20], then

$$\int e^{-i\varsigma\eta}\frac{\partial \omega}{\partial \tau}(\eta,\tau)d\eta-\frac{1}{2g\left(\hat{\lambda}\right)}\int e^{-i\varsigma\eta}\frac{\partial^2 \omega}{\partial \eta^2}(\eta,\tau)d\eta=\delta(\tau),$$

i.e.

$$\frac{\partial \hat{\omega}}{\partial \tau}+\frac{1}{2g\left(\hat{\lambda}\right)}|\varsigma|^2\hat{\omega}=\delta(\tau).$$

From the formula of the basic solution of the constant coefficients differential equation, we have

$$\hat{\omega}=e^{-\frac{1}{2g\left(\hat{\lambda}\right)}|\varsigma|^2\tau}\left(H(\tau)+C(\varsigma)\right).$$

where $H(\cdot)$ is a Heaviside function.

Since ω is a tempered distribution, $C(\varsigma)=0$, and

$$\hat{\omega}=H(\tau)e^{-\frac{1}{2g\left(\hat{\lambda}\right)}|\varsigma|^2\tau}.$$

Hence

$$\omega(\eta,\tau)=\frac{1}{2\pi}\int e^{i\varsigma\eta}\hat{\omega}(\varsigma,\tau)d\varsigma=\frac{H(\tau)}{2\pi}\int e^{i\varsigma\eta}e^{-\frac{1}{2g\left(\hat{\lambda}\right)}|\varsigma|^2\tau}d\varsigma.$$

Using the Fourier transformation formula of Gauss function, we have

$$\omega(\eta,\tau)=\frac{H(\tau)}{2\pi}\int e^{-i\vartheta\left(-\frac{\eta}{\sqrt{2\tau}}\right)}e^{-\frac{1}{4g\left(\hat{\lambda}\right)}|\vartheta|^2}\frac{1}{2\tau}d\vartheta \quad \text{(substitution } \varsigma=\frac{\vartheta}{\sqrt{2\tau}}\text{)}$$

$$=\frac{H(\tau)}{2\pi}\frac{1}{(2\tau)^{\frac{1}{2}}}(2\pi)^{\frac{1}{2}}\left(2g\left(\hat{\lambda}\right)\right)^{\frac{1}{2}}e^{-g\left(\hat{\lambda}\right)\left|\frac{\eta}{\sqrt{2\tau}}\right|^2}$$

$$=H(\tau)\sqrt{\frac{g\left(\hat{\lambda}\right)}{2\pi\tau}}e^{-g\left(\hat{\lambda}\right)\frac{\eta^2}{2\tau}}.$$

Hence

$$\omega(\eta,\tau)=\begin{cases}0, & t\le 0,\\ \sqrt{\dfrac{g(\hat{\lambda})}{2\pi\tau}}e^{-g(\hat{\lambda})\frac{\eta^2}{2\tau}}, & t>0.\end{cases}$$

Since the heat-conduction equation is an irreversible process, we only consider the Cauchy problem of $t \ge 0$.

Let $Z(\eta-\zeta,\tau;y,\kappa)$ be the fundamental solution of the equation

$$\frac{\partial \omega}{\partial \tau}=\frac{1}{2g(\hat{\lambda})}\frac{\partial^2 \omega}{\partial \eta^2},$$

i.e.

$$Z(\eta-\zeta,\tau;y,\kappa)=\sqrt{\frac{g(\hat{\lambda})}{2\pi(\tau-\kappa)}}\exp\left[-\frac{g(\hat{\lambda})(\eta-\zeta)^2}{2(\tau-\kappa)}\right].$$

Define

$$K(\eta,\tau;\zeta,\kappa)=\left[\frac{1}{2g(\bar{\lambda})}\frac{\partial^2}{\partial\eta^2}-\frac{\sigma}{2g(\bar{\lambda})}\left(\frac{T}{\sqrt[3]{3T^2\tau}}\right)\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\tau}\right]Z(\eta-\zeta,\tau;\zeta,\kappa),$$

$$\Phi(\eta,\tau;\zeta,\kappa)=\sum_{m=1}^{\infty}K_m(\eta,\tau;\zeta,\kappa),$$

where $K_1=K$, $K_m(\eta,\tau;\zeta,\kappa)=\int_\kappa^\tau d\theta\int_R K_1(\eta,\tau;y,\theta)K_{m-1}(y,\theta;\zeta,\kappa)dy,(m\ge 2)$.

The fundamental solution Ψ of Equation (4.6) is

$$\Psi(\eta,\tau;\zeta,\kappa)=Z(x-\zeta,\tau;\zeta,\kappa)+\int_\kappa^\tau d\theta\int_R Z(x-y,\tau;y,\theta)\Phi(y,\theta;\zeta,\tau)dy.$$

Hence the solution of Cauchy problem (4.5) is

$$W(\eta,\tau)=\int_R\Psi(\eta,\tau;\rho,0)\times\left(e^{\sigma\rho}-K\right)^+d\rho.$$

Return to the original variables, the semi-explicit solution of the equation (3.11) can be obtained as follows

$$V(S,J,t)=\exp\left(-r\left(T-\frac{(T-t)^3}{3T^2}\right)\right)\int_R\Psi\left(\frac{(T-t)^3}{3T^2},\frac{t\ln J+(T-t)\ln S}{\sigma T}+\frac{r(T-t)^2}{2\sigma T};0,\rho\right)\times\left(e^{\sigma\rho}-K\right)^+d\rho.$$

V. SUMMARY

The main work of this paper is to do further research on the basis of predecessors. This paper mainly studies the pricing model of Asian option on Riemannian manifold, and gives a semi-explicit solution to the pricing formula of geometric average Asian call option with fixed strike price.

With the development of market economy, there are many kinds of options, such as lookback options, knock-out options and so on. In order to be closer to the real financial market, we can consider the research of all kinds of option pricing problems under the non-risk neutral measurement , and the research of all kinds of option pricing problems under the incomplete hedging market in the future research of option pricing.

REFERENCES

[1] Bernt Øksendal. Stochastic Differential Equations[M]. Springer. 2006.
[2] Elton P. Hsu. Stochastic Analysis On Manifolds[M]. American Mathematical Society. Providence, RI, 2002.
[3] Yong-Chao Zhang. Option Pricing Under General Geometric Riemannian Brownian Motions[J]. Bull, Korean Math. Soc. 53(2016). No. 5:1411-1425.
[4] J.Hull , A.White. The Pricing of Options on Assets with Stochastic Volatility[J]. J. Finance. 42(1987). No.2, 281-300.
[5] A.Friedman. Stochastic Differential Equations and Applications[M]. Academic Press. 1976.
[6] A.Friedman. Partial Differential Equations of Parabolic Type[M]. Prentice-Hall. 1964. 288-322.
[7] F. Black & M. Scholes. The Pricing of Options and Corporate Liabilities[J]. Journal of Political Economy, 1973, 81(3):637-654.
[8] P.J.Hunt, J.E.Kennendy. Financial Derivatives in Theory and Practice[M]. Revised ed. John Wiley & Sons, Ltd. 2004.
[9] S.E.Shreve. Stochastic Calculus for Finance[M]. Springer-Verlag. New York Inc. 2004.
[10] I. Karatzas, S.E.Shreve. Browwnian Motion and Stochastic Calculus[M], 2 ed. Springer-Verlag. New York. 1991.
[11] Chan T. Pricing contingent claims on stocks driven by Lévy processes[J]. The Annals of Applied Probability, 1999, 9(2):504-528.
[12] H.X. Wu, C.L. Shen & Y.L. Yu. Preliminary study of Riemann geometry[M]. Beijing: Higher Education Press, 2014.
[13] H.W. Zou. Principle of asset pricing[M]. Economic Science Press, 2010.
[14] Stein E M, Stein J C. Stock Price Distributions with Stochastic Volatility: An Analytic Approach[J]. Review of Financial Studies, 1991, 4(4):727-752.
[15] X. Lai & Q.C. Feng. Summary of Asian option pricing research[J]. Modern Business Industry, 2009,21(24):170-172.
[16] Chalasani P, Jha S, Varikooty A. Accurate Approximations for European-Style Asian Options[J]. Journal of Computational Finance, 1998, 1(4):11-30.
[17] D.Z. Guo. Mathematical Finance: the principle and model of asset pricing [M]. Tsinghua University Press, 2006.
[18] Merton R C. Option pricing when underlying stock returns are discontinuous[J]. Working papers, 1975, 3(1):125-144.
[19] Zhang J E. A Semi-Analytical Method for Pricing and Hedging Continuously Sampled Arithmetic Average Rate Options[J]. Journal of Computational Finance, 2001, 5(1):59-79.
[20] Reed M, Simon B. Methods of modern mathematical physics[M]. World Book Publishing Company. 2003.
[21] L.S. Jiang. Mathematical modeling and methods of option pricing[M]. Beijing:Higher Education Press, 2008.
[22] Z.Z. Tang. Fundamentals of Riemann geometry[M]. Beijing:Beijing Normal University Press, 2011.
[23] Edmond Lévy. Pricing European average rate currency options[M]. Journal of International Money and Finance. 1992,10:261-287.
[24] Stein E M, Stein J C. Stock Price Distributions with Stochastic Volatility: An Analytic Approach[J]. Review of Financial Studies, 1991, 4(4):727-752.
[25] Nengjiu Ju. Pricing Asian and Basker Options via Taylor Expansion[J]. The Journal of Computationl Finance. 2002,5(3):79-103.
[26] Kemna, A. G. Z, Vorst, A. C. F. A pricing method for options based on average asset values[J]. Journal of Banking & Finance, 1990, 14(1):113-129.